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复 Finsler 流形间的调和映射

肖金秀[1]，贺　群[2]，陈志华[2]，邱春晖[3]

(1. 上海工程技术大学 高职学院，上海 200437；2. 同济大学 数学系，上海 200092；3. 厦门大学 数学科学学院，福建 厦门 361005)

摘要：通过定义其上的整体内积得到相应 的伴随算子和 Laplace 算子，并且通过计算得到了强拟凸复 Finsler 流形间光滑映射的∂-能量和$\bar{\partial}$-能量的变分公式，从而给出了调和映射的定义；最后得到∂-量与$\bar{\partial}$-量之差不是同伦不变的.

关键词：Laplace 算子；复 Finsler 度量；调和映射；∂-能量

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Harmonic Maps Between Complex Finsler Manifolds

XIAO Jinxiu[1]，*HE Qun*[2]，*CHEN Zhihua*[2]，*QIU Chunhui*[3]

(1. School of Advanced Vocational Education，Shanghai University of Engineering Science，Shanghai 200437，China；2. Department of Mathematics，Tongji University，Shanghai 200092，China；3. School of Mathematical Sciences，Xiamen University，Xiamen 361005，China)

Abstract：The variation formula of ∂-energy and $\bar{\partial}$-energy is obtained for a smooth map between strongly pseudoconvex complex Finsler manifolds. Also，for such maps，the difference between ∂-energy and $\bar{\partial}$-energy proves to be not a homotopy invariant on complex Finsler manifold.

Key words：Laplacian；complex Finsler metric；harmonic map；∂-energy

文献[1-5]给出了实 Finsler 流形间的调和映射的一些开创性成果. 而对于复 Finsler 流形的情形，基于比实 Finsler 流形更复杂，更不同于 Hermite 度量的实部即为 Riemann 度量，复 Finsler 度量的实部不再是实 Finsler 度量，导致复 Finsler 流形间的调和映射 的研究更复杂. 文献[6]通过考虑$\bar{\partial}$-能量变分研究了紧 Riemann 曲面到复 Finsler 流形上的调和映射. 最近，文献[7]则通过计算能量变分和应用文献[8] 中的非线性椭圆系统研究了复 Finsler 流形到 Hermite 流形上的调和映射，并且得到有关 Kähler Finsler 流形到复流形间调和映射的存在性定理.

本文，通过定义其上的整体内积得到相应 的伴随算子和 Laplace 算子，并且巧妙地给出了复 Finsler 度量和实 Finsler 度量之间的关系，得到了复 Finsler 流形间调和 映射的能量泛函与∂-能量泛函和$\bar{\partial}$-能量泛函之间的关系式，并且通过技巧性的计算分别得到了他们的变分公式，从而给出了调和映射的定义；由于复 Finsler 流形中有关复 Finsler 度量的联络系数不仅和底流形上的点有关而且和纤维也有关，进而导致∂-能量与$\bar{\partial}$-能量之差不是同伦不变的.

1　预备知识

设 M 为复 n 维的复流形，(z^k)为其局部坐标，$T^{1,0}M$ 为其全纯切丛. 设 $u=(z^k,\eta^k)\in T^{1,0}M$，$F(u)$为强拟凸复 Finsler 度量，由 $z^k=x^k+\mathrm{i}x^{n+k}$ 和 $\eta^k=y^k+\mathrm{i}y^{n+k}$ 知，实函数 $F(u)=\overset{R}{F}(x^a,y^b)$ 不再是 $T_RM\backslash\{0\}$上实 Finsler 度量. 若$(g_{j\bar{k}})$ 确定的矩阵 $\overset{R}{\boldsymbol{g}}_{ab}$ 是正定的，则称 $\boldsymbol{g}_{j\bar{k}}$ 是强凸的[9]. 因此，Munteanu 引进不同于 Abate 和 Patrizio 的方法，实的度量结构不是由 $\overset{R}{F}$ 决定，而是由矩阵 $\boldsymbol{g}_{j\bar{k}}$ 的实部确定.

命题 1[9]　设 $g_{i\bar{j}}(z,\eta)$由强拟凸复 Finsler 度量 F 诱导 $T^{1,0}M$ 上的 Hermite 度量，则 $L^2(x,y):=g_{ab}(x,y)y^ay^b$ 为一实 Finsler 度量，其中 g

$$\overset{R}{g}_{jk}=\mathrm{Re}g_{j\bar{k}}=\frac{1}{2}(g_{j\bar{k}}+g_{k\bar{j}});$$

$$\overset{R}{g}_{n+jk}=-\mathrm{Img}_{j\bar{k}} \tag{1a}$$

$$\overset{R}{g}_{jn+k}=\mathrm{Img}_{j\bar{k}}=-\frac{i}{2}(g_{j\bar{k}}-g_{k\bar{j}});$$

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第一作者：肖金秀(1979—)，女，博士后，主要研究方向为多复变函数论与复几何. E-mail：xjx0502@163.com

$$\overset{R}{g}_{n+jn+k}=\mathrm{Re}g_{j\bar{k}} \tag{1b}$$

式中:Re 表示实部;Im 表示虚部.

称 L 为强拟凸复 Finsler 度量 F 诱导的实 Finsler 度量. 设 $\overset{R}{\boldsymbol{g}}{}^{ab}$ 为 $\overset{R}{\boldsymbol{g}}_{ab}$ 的逆矩阵,即为$(\overset{R}{g}{}^{ab})$,由 $g_{j\bar{k}}g^{l\bar{k}}=\delta_j^l$ 可得:

$$\overset{R}{g}{}^{jk}=\mathrm{Re}g^{k\bar{j}};\overset{R}{g}{}^{j(n+k)}=\mathrm{Im}g^{k\bar{j}};$$

$$\overset{R}{g}{}^{(n+j)(n+k)}=\mathrm{Re}g^{k\bar{j}};\overset{R}{g}{}^{(n+j)k}=\mathrm{Im}g^{k\bar{j}}; \tag{2}$$

设 $C^*=C\backslash\{0\}$,射影切丛 PTM 定义为 PTM$=\widetilde{M}/C^*$ 且 $\widetilde{\pi}$:PTM$\to M$,则 PTM 上的 Finsler 几何量关于切向量(即纤维坐标)是零齐次的. 令 $\mathrm{d}\tau=\sqrt{g}\,\mathrm{d}z^1\wedge\cdots\wedge\mathrm{d}z^n$,$\mathrm{d}\sigma=\frac{\sqrt{g}}{F^n}\sum_i(-1)^i\eta^i\delta\eta^1\wedge\cdots\wedge\delta\eta^i\wedge\cdots\wedge\delta\eta^n$,PTM 体积形式为 $\mathrm{d}V=\mathrm{d}\tau\wedge\mathrm{d}\bar{\tau}\wedge\mathrm{d}\sigma\wedge\mathrm{d}\bar{\sigma}$(参见[10-11]).

引理 1 设(M,F)为紧强拟凸复 Finsler 流形. 则对于所有射影切丛 PTM 上的函数 f,

$$\int_{\mathrm{PTM}}g^{k\bar{j}}[F^2f]_{\eta^k\bar{\eta}^j}\mathrm{d}V=$$

$$n\int_{\mathrm{PTM}}f\mathrm{d}V-\int_{\mathrm{PTM}}C^{\bar{j}}[F^2f]_{\bar{\eta}^j}\mathrm{d}V, \tag{3}$$

其中 $f_{\eta^i\bar{\eta}^j}=\frac{\partial^2 f}{\partial\eta^i\partial\bar{\eta}^j}$, $C^{\bar{j}}=g^{j(n+k)}C_{kl}^l$.

证明 设 $\phi=g^{k\bar{j}}[F^2f]_{\bar{\eta}^j}\left[i\left(\frac{\partial}{\partial\eta^k}\right)\mathrm{d}V\right]$,其中 $\mathrm{i}\left(\frac{\partial}{\partial\eta^k}\right)$为内乘积,$\mathrm{d}\xi=\sum_i(-1)^i\eta^i\delta\eta^1\wedge\cdots\wedge\widehat{\delta\eta^i}\wedge\cdots\wedge\delta\eta^n$. 则

$$\delta\eta^l\wedge\frac{\partial}{\partial\eta^l}[i(\frac{\partial}{\partial\eta^k})\mathrm{d}V]=\frac{\partial}{\partial\eta^k}(\frac{g^2}{F^{2n}})\mathrm{d}z\wedge\mathrm{d}\bar{z}\wedge$$

$$\mathrm{d}\xi\wedge\mathrm{d}\bar{\xi}=(2C_{kl}^l-n\frac{g_{k\bar{l}}\bar{\eta}^l}{F^2})\mathrm{d}V.$$

因此,

$$\mathrm{d}\phi=g^{k\bar{j}}[F^2f]_{\eta^k\bar{\eta}^j}\mathrm{d}V-nf\mathrm{d}V+g^{k\bar{j}}[F^2f]_{\bar{\eta}^j}C_{kl}^l\mathrm{d}V.$$

在 PTM 上积分得

$$\int_{\mathrm{PTM}}g^{k\bar{j}}[F^2f]_{\eta^k\bar{\eta}^j}\mathrm{d}V=n\int_{\mathrm{PTM}}f\mathrm{d}V-$$

$$\int_{\mathrm{PTM}}g^{k\bar{j}}C_{kl}^l[F^2f]_{\bar{\eta}^j}\mathrm{d}V.$$

若令 $\phi=g^{k\bar{j}}[F^2f]_{\eta^k}[\mathrm{i}(\frac{\partial}{\partial\bar{\eta}^j})\mathrm{d}V]$,同样可得

$$\int_{\mathrm{PTM}}g^{k\bar{j}}[F^2f]_{\eta^k\bar{\eta}^j}\mathrm{d}V=n\int_{\mathrm{PTM}}f\mathrm{d}V-$$

$$\int_{\mathrm{PTM}}C^j[F^2f]_{\eta^j}\mathrm{d}V,$$

由式(3)得

$$\int_{\mathrm{PTM}}C^{\bar{j}}[F^2f]_{\bar{\eta}^j}\mathrm{d}V=\int_{\mathrm{PTM}}C^j[F^2f]_{\eta^j}\mathrm{d}V. \tag{4}$$

引理 2 设(M,F)为紧强拟凸复 Finsler 流形. 若 $T^{1,0}M$ 上的光滑函数 f 满足 $f(z,\lambda\eta)=\bar{\lambda}f(z,\eta)$,则

$$\int_{\mathrm{PTM}}\frac{1}{F^2}\eta^k\frac{\delta f}{\delta z^k}\mathrm{d}V=$$

$$\int_{\mathrm{PTM}}\frac{1}{F^2}f(L_{kj}^j-L_{jk}^j)\eta^k\mathrm{d}V. \tag{5}$$

证明 令 $\chi=\frac{1}{F^2}\eta^kfi\left(\frac{\delta}{\delta z^k}\right)\mathrm{d}V$,而$\frac{\delta}{\delta z^k}(F)=0$,从而有

$$\mathrm{d}\chi=\frac{1}{F^2}\left\{\frac{\delta f}{\delta z^k}+f(L_{jk}^j-L_{kj}^j)\right\}\eta^k\mathrm{d}V,$$

因此

$$\int_{\mathrm{PTM}}\frac{1}{F^2}\left\{\frac{\delta f}{\delta z^k}+f(L_{jk}^j-L_{kj}^j)\right\}\eta^k\mathrm{d}V=0.$$

2 调和映射

设(M,F)为 n 维紧强拟凸复 Finsler 流形,(N,L)为 m 维强拟凸复 Finsler 流形. 设 $f:M\to N$ 为非退化光滑映射. 记 M 的局部坐标为$\{z^i\}$以及$\{\omega^\mu\}$为 N 的局部坐标,f 局部可表示为

$$\omega^\kappa=f^\mu(z^1,\cdots,z^n,\bar{z}^1,\cdots,\bar{z}^n),\ 1\leqslant\mu,\nu\leqslant m.$$

设有关度量 F 的 Chern-Finsler 联络的联络系数为 $\Gamma_{jk}^i=h^{\bar{l}i}\frac{\delta h_{j\bar{l}}}{\delta z^k}$, $C_{jk}^i=h^{\bar{l}i}\frac{\partial h_j\bar{l}}{\partial\eta^k}$,$\Gamma_j^i=h^{\bar{l}i}\frac{\partial^2F^2}{\partial z^j\partial\bar{\eta}^l}$;有关度量 L 的联络的联络系数为 $L_{\beta\gamma}^\alpha=g^{\bar{\delta}\alpha}\frac{\delta g_{\beta\bar{\delta}}}{\delta\omega^\gamma}$,$C_{\beta\gamma}^\alpha=g^{\bar{\delta}\alpha}\frac{\partial g_{\beta\bar{\delta}}}{\partial\bar{\eta}^\gamma}$,$N_\beta^\alpha=g^{\bar{\delta}\alpha}\frac{\partial^2L^2}{\partial\omega^\beta\partial\bar{\eta}^\delta}$. 设 $\eta=\eta^i\frac{\partial}{\partial z^i}+\bar{\eta}^i\frac{\partial}{\partial\bar{z}^i}$,则 $\mathrm{d}f(\eta)=\eta^jf_i^\mu\frac{\partial}{\partial\omega^\mu}+\eta^jf_i^{\bar{\nu}}\frac{\partial}{\partial\bar{\omega}^\nu}+\bar{\eta}^if_{\bar{i}}^\mu\frac{\partial}{\partial\omega^\mu}+\bar{\eta}^i\bar{f}_{\bar{i}}^{\bar{\nu}}\frac{\partial}{\partial\bar{\omega}^\nu}$,其中 $f_i^\mu=\frac{\partial f^\mu}{\partial z^i}$,$f_{\bar{i}}^\mu=\frac{\partial f^\mu}{\partial\bar{z}^i}$,$f_i^{\bar{\mu}}=\frac{\partial\bar{f}^\mu}{\partial z^i}$. 若设 $\zeta=\zeta^\mu\frac{\partial}{\partial\omega^\mu}$,$\hat{\zeta}=\hat{\zeta}^\nu\frac{\partial}{\partial\omega^\nu}$,其中 $\zeta^\mu=\eta^if_i^\mu$,$\hat{\zeta}^\nu=\bar{\eta}^if_{\bar{i}}^\kappa$. 则 f 的∂—能量密度和$\bar{\partial}$—能量密度可分别定义为

$$e'(f)=|\partial f|^2=h^{\bar{i}\bar{j}}f_i^\mu\bar{f}_{\bar{j}}^\nu g_{\mu\bar{\nu}}(f(z),\zeta), \tag{6a}$$

$$e''(f)=|\bar{\partial}f|^2=h^{\bar{i}\bar{j}}f_{\bar{j}}^{\bar{\nu}}f_{\bar{i}}^\mu g_{\mu\bar{\nu}}(f(z),\hat{\zeta}), \tag{6b}$$

其中 f 既非全纯也非反全纯. 否则,若 f 全纯则 $e''(f)=0$;若 f 反全纯则 $e'(f)=0$. f 的∂—能量泛函和$\bar{\partial}$—的能量泛函可定义为

$$E'(f)=\int_{\mathrm{PTM}}e'(f)\mathrm{d}V,\ E''(f)=$$

$$\int_{\mathrm{PTM}}e''(f)\mathrm{d}V, \tag{7}$$

其中 $h=\det(\boldsymbol{h}_{i\bar{j}})$.

由引理 2,公式(6)可重写为

$$e'(f)=|\partial f|^2=h^{i\bar{j}}[L^2(f(z),\zeta)]_{\eta^i\bar{\eta}^j},\quad(8a)$$

$$e''(f)=|\bar{\partial} f|^2=h^{i\bar{j}}[L^2(f(z),\hat{\zeta})]_{\eta^i\bar{\eta}^j},\quad(8b)$$

从而式(7)也可重写为

$$E'(f)=n\int_{PTM}\frac{L^2(f(z),\zeta)}{F^2(z,\eta)}\mathrm{d}V-\int_{PTM}C^{\bar{j}}[L^2(f(z),\zeta)]_{\bar{\eta}^j}\mathrm{d}V,\quad(9a)$$

$$E''(f)=n\int_{PTM}\frac{L^2(f(z),\hat{\zeta})}{F^2(z,\eta)}\mathrm{d}V-\int_{PTM}C^{\bar{j}}[L^2(f(z),\hat{\zeta})]_{\bar{\eta}^j}\mathrm{d}V,\quad(9b)$$

记 T_RM 的实局部坐标为$(x^1,\cdots,x^{2n},y^1,\cdots,y^{2n})$及 T_RN 的实局部坐标为$(\tilde{x}^1,\cdots,\tilde{x}^{2m},\tilde{y}^1,\cdots,\tilde{y}^{2m})$,其中 $z^i=x^i+\mathrm{i}x^{n+i}$,$\omega^\mu=\tilde{x}^\mu+\mathrm{i}\,\tilde{x}^{m+\mu}$,则

$$\mathrm{d}f=\frac{\partial\tilde{x}^a}{\partial x^b}\frac{\partial}{\partial\tilde{x}^a}\otimes\mathrm{d}x^b,$$

其中 $1\leqslant a,\cdots\leqslant 2m,1\leqslant b,\cdots\leqslant 2n$ 且

$$|\mathrm{d}f|^2=\overset{R}{h}{}^{bd}\frac{\partial\tilde{x}^a}{\partial x^b}\frac{\partial\tilde{x}^c}{\partial x^d}g^R_{ac},\quad(10)$$

$1\leqslant a,c\cdots\leqslant 2m,1\leqslant b,d\cdots\leqslant 2n$. 根据式(1)—(2),得

$$|\mathrm{d}f|^2=2|\partial f|^2+2|\bar{\partial}f|^2.\quad(11)$$

从而,f 的能量密度可定义为 $e(f)=\frac{1}{2}|\mathrm{d}f|^2$. 由式(6),(11),有

$$e(f)=e'(f)+e''(f),\quad(12)$$

且

$$\int_{PTM}e(f)\mathrm{d}V=\int_{PTM}e'(f)\mathrm{d}V+\int_{PTM}e''(f)\mathrm{d}V.\quad(13)$$

则,

$$E(f)=E'(f)+E''(f).\quad(14)$$

现考虑 $f=f_0$ 的光滑变分,即一族光滑映射

$$f_t:M\to N,\quad t\in D=\{z\in C\,|\,|z|<\varepsilon\}.$$

则 f 的∂-能量泛函和$\bar{\partial}$-能量泛函变分为

$$\frac{\partial}{\partial t}(E'(f_t))|_{t=0}=n\int_{PTM}\frac{\partial}{\partial t}\frac{L^2(f_t(z),\zeta_t)}{F^2(z,\eta)}|_{t=0}\mathrm{d}V-\int_{PTM}C^{\bar{j}}\frac{\partial}{\partial t}[L^2(f_t(z),\zeta_t)]_{\bar{\eta}^j}|_{t=0}\mathrm{d}V,\quad(15a)$$

$$\frac{\partial}{\partial t}(E''(f_t))|_{t=0}=n\int_{PTM}\frac{\partial}{\partial t}\frac{L^2(f_t(z),\hat{\zeta}_t)}{F^2(z,\eta)}|_{t=0}\mathrm{d}V-\int_{PTM}C^{\bar{j}}\frac{\partial}{\partial t}[L^2(f_t(z),\hat{\zeta}_t)]_{\bar{\eta}^j}|_{t=0}\mathrm{d}V,\quad(15b)$$

其中 $\zeta_0=\zeta=\eta^if^\mu_i\frac{\partial}{\partial\omega^\mu}$,　$\hat{\zeta}_0=\hat{\zeta}=\bar{\eta}^if^v_{\bar{i}}\frac{\partial}{\partial\omega^v}$.

由变分$\{f_t\}$诱导 $f_t^{-1}T_CN$ 上的向量为

$$\mathbf{V}:=\mathrm{d}f_t(\frac{\partial}{\partial t})=\partial f_t(\frac{\partial}{\partial t})+\partial\bar{f}_t(\frac{\partial}{\partial t})=V'_t+V''_t,\quad(16)$$

且

$$V'_0=V'=\frac{\partial f^\alpha_t}{\partial t}|_{t=0}\frac{\partial}{\partial\omega^\alpha},$$

$$V''_0=V''=\frac{\partial\bar{f}^\alpha_t}{\partial t}|_{t=0}\frac{\partial}{\partial\bar{\omega}^\alpha}$$

则

$$\frac{\partial}{\partial t}L^2(f_t(z),\zeta_t)_{t=0}=\{[V'^\alpha\frac{\partial}{\partial\omega^\alpha}+\eta^i\frac{\delta V'^\alpha}{\delta z^i}\frac{\partial}{\partial\zeta^\alpha}]+[V''^\alpha\frac{\partial}{\partial\bar{\omega}^\alpha}+\bar{\eta}^i\frac{\delta V''^\alpha}{\delta\bar{z}^i}\frac{\partial}{\partial\bar{\zeta}^\alpha}]\}L^2(f(z),\zeta).$$

由引理 2 得

$$\int_{PTM}\frac{1}{F^2}\eta^i\frac{\delta}{\delta z^i}(V'^\alpha)\frac{\partial}{\partial\zeta^\alpha}L^2(f(z),\zeta)\mathrm{d}V=-\int_{PTM}\frac{1}{F^2}V'^\alpha\eta^i\frac{\delta}{\delta z^i}(\frac{\partial}{\partial\zeta^\alpha}L^2(f(z),\zeta))\mathrm{d}V+\int_{PTM}\frac{1}{F^2}V'^\alpha\frac{\partial}{\partial\zeta^\alpha}L^2(f(z),\zeta)(\Gamma^k_{ik}-\Gamma^k_{ki})\eta^i\mathrm{d}V,$$

且

$$\int_{PTM}\frac{1}{F^2}[V'^\alpha\frac{\partial}{\partial\omega^\alpha}+\eta^i\frac{\delta V'^\alpha}{\delta z^i}\frac{\partial}{\partial\zeta^\alpha}]L^2(f(z),\zeta)\mathrm{d}V=\int_{PTM}\frac{1}{F^2}\Big\{g_{\varepsilon\bar{\beta}}(L^\varepsilon_{\gamma\alpha}-L^\varepsilon_{\alpha\gamma})\zeta^\gamma\bar{\zeta}^\beta-g_{\alpha\bar{\beta}}(\eta^i\bar{\eta}^j\bar{f}^\beta_{j\bar{i}}+\bar{N}^\beta_\gamma\bar{\hat{\zeta}}^\gamma)-(\eta^i\eta^kf^\sigma_{ik}+N^\sigma_\gamma\zeta^\gamma-\eta^i\Gamma^k_if^\sigma_k)g_{\alpha\bar{\beta}\sigma}\bar{\zeta}^\beta+g_{\alpha\bar{\beta}}\bar{\zeta}^\beta(\Gamma^k_{ik}-\Gamma^k_{ki})\eta^i\Big\}V'^\alpha\mathrm{d}V.$$

类似地,

$$\int_{PTM}[V'^\alpha\frac{\partial}{\partial\omega^\alpha}+\eta^i\frac{\delta V'^\alpha}{\delta z^i}\frac{\partial}{\partial\zeta^\alpha}](C^{\bar{j}}\bar{f}^v_jg_{\mu\bar{v}}\zeta^\mu)\mathrm{d}V=\int_{PTM}\Big\{C^{\bar{j}}\{\bar{f}^\beta_j(L^\gamma_{\mu\alpha}-L^\gamma_{\alpha\mu})\zeta^\mu g_{\gamma\bar{\beta}}-g_{\alpha\bar{\beta}}(\eta^i\bar{f}^\beta_{j\bar{i}}+\bar{\hat{\zeta}}^\sigma\bar{L}^\beta_{\gamma\sigma}\bar{f}^\gamma_j)-\bar{f}^\beta_j(\eta^i\bar{\eta}^l\bar{f}^\gamma_{l\bar{i}}+\bar{\hat{\zeta}}^\sigma\bar{N}^\gamma_\sigma)g_{\alpha\bar{\beta}\bar{\gamma}}-\bar{f}^\beta_j(\eta^i\eta^kf^\gamma_{ki}+\zeta^\mu N^\gamma_\mu-\eta^i\Gamma^k_if^\gamma_\kappa)g_{\alpha\bar{\beta}\gamma}+\bar{f}^\beta_jg_{\alpha\bar{\beta}}(\Gamma^k_{ik}-\Gamma^k_{ki})\eta^i\}-\eta^ig_{\alpha\bar{\beta}}\bar{f}^\beta_jg^{k\bar{j}}\partial_k(\Gamma^l_{li})\Big\}V'^\alpha\mathrm{d}V,\int_{PTM}\frac{1}{F^2}[V''^\beta\frac{\partial}{\partial\bar{\omega}^\beta}+\bar{\eta}^j\frac{\delta V''^\beta}{\delta\bar{z}^j}\frac{\partial}{\partial\bar{\zeta}^\beta}]\cdot L^2(f(z),\zeta)\mathrm{d}V=\int_{PTM}\frac{1}{F^2}\Big\{g_{\alpha\bar{\varepsilon}}(\bar{L}^\varepsilon_{\gamma\beta}-\bar{L}^\varepsilon_{\beta\gamma})\zeta^\alpha\bar{\zeta}^\gamma-g_{\alpha\bar{\beta}}(\eta^i\bar{\eta}^jf^\alpha_{i\bar{j}}+N^\alpha_\gamma\hat{\zeta}^\gamma)-(\bar{\eta}^i\bar{\eta}^k\bar{f}^\sigma_{ik}+\bar{N}^\sigma_\gamma\bar{\zeta}^\gamma-\bar{\eta}^i\bar{\Gamma}^k_i\cdot\bar{f}^\sigma_k)g_{\alpha\bar{\beta}\bar{\sigma}}\zeta^\alpha+g_{\alpha\bar{\beta}}\zeta^\alpha(\bar{\Gamma}^k_{ik}-\bar{\Gamma}^k_{ki})\bar{\eta}^i\Big\}V''^\beta\mathrm{d}V.$$

以及

$$\int_{PTM}[V''^\beta\frac{\partial}{\partial\bar{\omega}^\beta}+\bar{\eta}^k\frac{\delta V''^\beta}{\delta\bar{z}^k}\frac{\partial}{\partial\bar{\zeta}^\beta}](C^{\bar{j}}\bar{f}^v_jg_{\mu\bar{v}}\zeta^\mu)\mathrm{d}V=\int_{PTM}\Big\{C^{\bar{j}}\{g_{\alpha\bar{\gamma}}(\bar{L}^\gamma_{\sigma\beta}-\bar{L}^\gamma_{\beta\sigma})\bar{\zeta}^\alpha f^\sigma_j+g_{\alpha\bar{\varepsilon}\bar{\gamma}}(\bar{L}^\varepsilon_{\sigma\beta}-\bar{L}^\varepsilon_{\beta\sigma})\bar{\zeta}^\alpha\zeta^\sigma\bar{f}^\gamma_j+$$

$\bar{\eta}^l[\bar{\Gamma}^k_{lk}-\bar{\Gamma}^k_{kl}]\zeta^\alpha\bar{f}^\gamma_j g_{\alpha\bar{\beta}\bar{\gamma}}-(\eta^k\bar{\eta}^l f^\alpha_{\bar{i}l}+\bar{\eta}^l f^\sigma_{\bar{l}}N^\alpha_\sigma-\bar{\eta}^l\cdot\bar{\Gamma}^h_{jl}\zeta^\alpha)\bar{f}^\gamma_j g_{\alpha\bar{\beta}\bar{\gamma}}-(\bar{\eta}^l\bar{f}^\gamma_{jl}+\bar{\zeta}^\sigma f^\varepsilon_j\bar{L}^\gamma_{\varepsilon\sigma})\zeta^\sigma g_{\alpha\bar{\beta}\bar{\gamma}}-\bar{\eta}^l\zeta^\alpha\bar{f}^\gamma_j\cdot[f^\sigma_{\bar{l}}g_{\alpha\bar{\varepsilon}}\dot{\partial}_{\bar{\gamma}}(\bar{L}^\varepsilon_{\beta\sigma})+f^\sigma_{\bar{l}}g_{\varepsilon\bar{\beta}}\dot{\partial}_{\bar{\gamma}}(L^\varepsilon_{\alpha\sigma})]-\bar{\eta}^l\zeta^\alpha\bar{f}^\gamma_j(\eta^k f^\sigma_{\bar{k}\bar{l}}+f^\varepsilon_{\bar{l}}N^\sigma_\varepsilon)g_{\alpha\bar{\beta}\sigma\bar{\gamma}}-\bar{\eta}^l\zeta^\alpha\bar{f}^\gamma_j(\bar{\eta}^k\bar{f}^\sigma_{kl}+\bar{f}^\varepsilon_l\bar{N}^\sigma_\varepsilon-\bar{\Gamma}^k_l\bar{f}^\sigma_k)g_{\alpha\bar{\beta}\bar{\sigma}\bar{\gamma}}-g_{\alpha\bar{\beta}}(\bar{\Gamma}^k_{jk}-\bar{\Gamma}^k_{kj})\zeta^\alpha-g_{\alpha\bar{\beta}}(\eta^i f^\alpha_{\bar{i}\bar{j}}+f^\sigma_{\bar{j}}N^\alpha_\sigma)-(\bar{\eta}^k\bar{f}^\sigma_{kj}+\bar{f}^\gamma_j\bar{N}^\sigma_\gamma-\bar{\Gamma}^k_j\bar{f}^\sigma_k)g_{\alpha\bar{\beta}\bar{\sigma}}\zeta^\alpha\}-\bar{\eta}^l[g^{k\bar{j}}g^{\iota\bar{\omega}}\Gamma^h_{\bar{k}l}g_{\mu\bar{h}\bar{\upsilon}}+g^{k\bar{j}}\dot{\partial}_k(\bar{\Gamma}^h_{hl})]\cdot g_{\alpha\bar{\beta}\bar{\gamma}}\zeta^\alpha\bar{f}^\gamma_j-[g^{k\bar{j}}g^{\iota\bar{\omega}}\Gamma^h_{\bar{k}j}g_{\mu\bar{h}\bar{\upsilon}}+g^{k\bar{j}}\dot{\partial}(\bar{\Gamma}^h_{hj})]g_{\alpha\bar{\beta}}\zeta^\alpha\Big\}\cdot V''^\beta\mathrm{d}V.$

因此,有

定理1 设$(M,\hat{F})$为紧强拟凸复Finsler流形,(N,L)为强拟凸复Finsler流形. 若$f:(M,F)\to(N,L)$为非退化且非反全纯映射,∂-能量泛函的第一变分为

$$\frac{\partial}{\partial t}(E'(f_t))\mid_{t=0}=\int_{PTM}\frac{1}{F^2}(n\rho_\alpha+F^2\gamma_\alpha)V'^\alpha\mathrm{d}V+\int_{PTM}\frac{1}{F^2}(n\bar{\rho}_\beta+F^2\tilde{\gamma}_\beta)V''^\beta\mathrm{d}V,\tag{17}$$

其中

$$\rho_\alpha=g_{\varepsilon\bar{\beta}}(L^\varepsilon_{\gamma\alpha}-L^\varepsilon_{\alpha\gamma})\zeta^\gamma\bar{\zeta}^\beta-g_{\alpha\bar{\beta}}(\eta^i\bar{\eta}^j\bar{f}^\beta_{\bar{j}\bar{i}}+\bar{N}^\beta_\gamma\bar{\zeta}^\gamma)-(\eta^i\eta^k f^\sigma_{ik}+N^\sigma_\gamma\zeta^\gamma-\eta^i\Gamma^k_i f^\sigma_k)g_{\alpha\bar{\beta}\sigma}\bar{\zeta}^\beta+g_{\alpha\bar{\beta}}\bar{\zeta}^\beta(\Gamma^k_{ik}-\Gamma^k_{ki})\eta^i,$$

以及

$$\gamma_\alpha=C^{\bar{j}}\{\bar{f}^\beta_j(L^\gamma_{\mu\alpha}-L^\gamma_{\alpha\mu})\zeta^\mu g_{\gamma\bar{\beta}}-g_{\alpha\bar{\beta}}(\eta^i\bar{f}^\beta_{\bar{j}\bar{i}}+\hat{\bar{\zeta}}^\sigma\bar{L}^\beta_{\gamma\sigma}\cdot\bar{f}^\gamma_j)-\bar{f}^\beta_j(\eta^i\bar{\eta}^l\bar{f}^\gamma_{\bar{l}\bar{i}}+\hat{\bar{\zeta}}^\sigma\bar{N}^\gamma_\sigma)g_{\alpha\bar{\beta}\bar{\gamma}}-\bar{f}^\beta_j(\eta^i\eta^k f^\gamma_{ki}+\zeta^\mu N^\gamma_\mu-\eta^i\Gamma^k_i f^\gamma_k)g_{\alpha\bar{\beta}\gamma}+\bar{f}^\beta_j g_{\alpha\bar{\beta}}(\Gamma^k_{ik}-\Gamma^k_{ki})\eta^i\}-\eta^i g_{\alpha\bar{\beta}}\bar{f}^\beta_j g^{k\bar{j}}\dot{\partial}_k(\Gamma^l_{li}),$$

和

$$\tilde{\gamma}_\beta=C^{\bar{j}}\{g_{\alpha\bar{\gamma}}(\bar{L}^\gamma_{\alpha\beta}-\bar{L}^\gamma_{\beta\alpha})\zeta^\alpha\bar{f}^\gamma_j+g_{\alpha\bar{\varepsilon}\bar{\gamma}}(\bar{L}^\varepsilon_{\alpha\beta}-\bar{L}^\varepsilon_{\beta\sigma})\zeta^\alpha\bar{\zeta}^\sigma\cdot\bar{f}^\gamma_j+\bar{\eta}^l[\bar{\Gamma}^k_{lk}-\bar{\Gamma}^k_{kl}]\zeta^\sigma\bar{f}^\gamma_j g_{\alpha\bar{\beta}\bar{\gamma}}-(\eta^i\bar{\eta}^l f^\alpha_{i\bar{l}}+\bar{\eta}^l f^\sigma_{\bar{l}}N^\alpha_\sigma)\cdot\bar{f}^\gamma_j g_{\alpha\bar{\beta}\bar{\gamma}}-(\bar{\eta}^l\bar{f}^\gamma_{jl}+\zeta^\sigma\bar{f}^\varepsilon_j\bar{L}^\gamma_{\varepsilon\sigma}-\bar{\eta}^l\bar{\Gamma}^h_{jl}\bar{f}^\gamma_h)\zeta^\sigma g_{\alpha\bar{\beta}\bar{\gamma}}-\bar{\eta}^l\cdot\zeta^\alpha\bar{f}^\gamma_j[f^\sigma_{\bar{l}}g_{\alpha\bar{\varepsilon}}\dot{\partial}_{\bar{\gamma}}(\bar{L}^\varepsilon_{\beta\sigma})+f^\sigma_{\bar{l}}g_{\varepsilon\bar{\beta}}\dot{\partial}_{\bar{\gamma}}(L^\varepsilon_{\alpha\sigma})]-\bar{\eta}^l\zeta^\alpha\bar{f}^\gamma_j(\eta^k f^\alpha_{\bar{k}\bar{l}}+f^\varepsilon_{\bar{l}}N^\sigma_\varepsilon)g_{\alpha\bar{\beta}\sigma\bar{\gamma}}-\bar{\eta}^l\zeta^\alpha\bar{f}^\gamma_j(\bar{\eta}^k\bar{f}^\sigma_{kl}+\bar{f}_l\bar{N}^\sigma_\varepsilon-\bar{\Gamma}^k_l\bar{f}^\sigma_k)g_{\alpha\bar{\beta}\bar{\sigma}\bar{\gamma}}-g_{\alpha\bar{\beta}}(\bar{\Gamma}^k_{jk}-\bar{\Gamma}^k_{kj})\zeta^\alpha-g_{\alpha\bar{\beta}}(\eta^i f^\alpha_{\bar{i}\bar{j}}+f^\sigma_{\bar{j}}N^\alpha_\sigma)-(\bar{\eta}^k\bar{f}^\sigma_{kj}+\bar{f}^\gamma_j\bar{N}^\sigma_\gamma-\bar{\Gamma}^k_j\bar{f}^\sigma_k)g_{\alpha\bar{\beta}\bar{\sigma}}\zeta^\alpha\}-\bar{\eta}^l[g^{\bar{k}j}g^{\iota\bar{\omega}}\Gamma^h_{\bar{k}l}g_{\mu\bar{h}\bar{\upsilon}}+g^{\bar{k}j}\dot{\partial}_k(\bar{\Gamma}^h_{hl})]g_{\alpha\bar{\beta}\bar{\gamma}}\zeta^\alpha\bar{f}^\gamma_j-[g^{k\bar{j}}g^{\iota\bar{\omega}}\Gamma^h_{\bar{k}j}g_{\mu\bar{h}\bar{\upsilon}}+g^{k\bar{j}}\dot{\partial}_k(\bar{\Gamma}^h_{hj})]g_{\alpha\bar{\beta}}\zeta^\alpha.$$

类似地,

定理2 设$(M,\hat{F})$为紧强拟凸复Finsler流形,(N,L)为强拟凸复Finsler流形. 若$f:(M,F)\to(N,L)$为非退化且非全纯映射. 则$\bar{\partial}$-能量泛函的第一变分为

$$\frac{\partial}{\partial t}(E''(f_t))\mid_{t=0}=\int_{PTM}\frac{1}{F^2}(n\hat{\rho}_\alpha+F^2\hat{\gamma}_\alpha)V'^\alpha\mathrm{d}V+\int_{PTM}\frac{1}{F^2}(n\hat{\bar{\rho}}_\beta+F^2\theta_\beta)V''^\beta\mathrm{d}V,\tag{18}$$

其中:

$$\hat{\rho}_\alpha=g_{\varepsilon\bar{\beta}}(L^\varepsilon_{\gamma\alpha}-L^\varepsilon_{\alpha\gamma})\hat{\zeta}^\gamma\hat{\bar{\zeta}}^\beta-g_{\alpha\bar{\beta}}(\eta^i\bar{\eta}^j\bar{f}^\beta_{\bar{j}i}+\bar{N}^\beta_\gamma\bar{\zeta}^\gamma)-(\eta^i\bar{\eta}^k f^\sigma_{i\bar{k}}+N^\sigma_\gamma\hat{\zeta}^\gamma-\bar{\eta}^i\Gamma^k_i f^\sigma_{\bar{k}})g_{\alpha\bar{\beta}\sigma}\hat{\bar{\zeta}}^\beta+g_{\alpha\bar{\beta}}\hat{\bar{\zeta}}^\beta(\bar{\Gamma}^k_{ik}-\bar{\Gamma}^k_{ki})\bar{\eta}^i,$$

以及

$$\vartheta_\beta=C^{\bar{j}}\{f^\alpha_{\bar{j}}(\bar{L}^\gamma_{\alpha\beta}-\bar{L}^\gamma_{\beta\alpha})\hat{\bar{\zeta}}^\sigma g_{\alpha\bar{\gamma}}-g_{\alpha\bar{\beta}}(\eta^i f^\alpha_{\bar{j}i}+\zeta^\sigma L^\alpha_{\gamma\sigma}f^\gamma_{\bar{j}})-\bar{f}^\beta_j(\eta^i\bar{\eta}^l\bar{f}^\gamma_{\bar{l}\bar{i}}+\hat{\bar{\zeta}}^\sigma\bar{N}^\gamma_\sigma)g_{\alpha\bar{\beta}\bar{\gamma}}-f^\alpha_{\bar{j}}(\eta^i\eta^k f^\sigma_{\bar{k}i}+\hat{\bar{\zeta}}^\gamma\bar{N}^\sigma_\gamma-\eta^i\Gamma^k_i f^\sigma_{\bar{k}})g_{\alpha\bar{\beta}\bar{\sigma}}+f^\alpha_{\bar{j}}g_{\alpha\bar{\beta}}(\Gamma^k_{ik}-\Gamma^k_{ki})\eta^i\}-\eta^i g_{\alpha\bar{\beta}}f^\alpha_{\bar{j}}g^{k\bar{j}}\dot{\partial}_k(\Gamma^l_{li}),$$

和

$$\hat{\gamma}_\alpha=C^{\bar{j}}\{g_{\gamma\bar{\beta}}(L^\gamma_{\alpha\alpha}-L^\gamma_{\alpha\sigma})\hat{\bar{\zeta}}^\beta f^\sigma_{\bar{j}}+g_{\varepsilon\bar{\beta}\gamma}(L^\varepsilon_{\alpha\alpha}-L^\varepsilon_{\alpha\sigma})\zeta^\sigma\hat{\bar{\zeta}}^\beta\cdot f^\gamma_{\bar{j}}+\bar{\eta}^l[\bar{\Gamma}^k_{lk}-\bar{\Gamma}^k_{kl}]\hat{\bar{\zeta}}^\beta f^\gamma_{\bar{j}}g_{\alpha\bar{\beta}\gamma}-(\bar{\eta}^l f^\gamma_{\bar{j}\bar{l}}\hat{\bar{\zeta}}^\beta+\bar{\zeta}^\sigma f^\gamma_{\bar{j}}\bar{N}^\beta_\sigma)g_{\alpha\bar{\beta}\gamma}-(\bar{\eta}^l f^\gamma_{\bar{j}\bar{l}}+\hat{\zeta}^\sigma f^\varepsilon_{\bar{j}}L^\gamma_{\varepsilon\sigma}-\bar{\eta}^l\bar{\Gamma}^h_{jl}f^\gamma_{\bar{h}})\hat{\bar{\zeta}}^\beta g_{\alpha\bar{\beta}\gamma}-\hat{\bar{\zeta}}^\beta f^\gamma_{\bar{j}}[\hat{\zeta}^\sigma g_{\varepsilon\bar{\beta}}\dot{\partial}_\alpha(L^\varepsilon_{\gamma\sigma})+\bar{\zeta}^\sigma g_{\varepsilon\bar{\varepsilon}}\dot{\partial}_\alpha(\bar{L}^\varepsilon_{\beta\sigma})]-\bar{\eta}^k\hat{\bar{\zeta}}^\beta f^\gamma_{\bar{j}}(\bar{\eta}^l f^\sigma_{\bar{k}\bar{l}}+\bar{f}^\varepsilon_k\bar{N}^\sigma_\varepsilon)g_{\alpha\bar{\beta}\gamma\bar{\sigma}}-\bar{\eta}^k\hat{\bar{\zeta}}^\beta f^\gamma_{\bar{j}}(\bar{\eta}^l f^\sigma_{\bar{k}\bar{l}}+f^\varepsilon_{\bar{k}}\bar{N}^\sigma_\varepsilon-\bar{\Gamma}^\mu_k f^\sigma_{\bar{\mu}})g_{\alpha\bar{\beta}\sigma\gamma}-g_{\alpha\bar{\beta}}(\bar{\Gamma}^k_{jk}-\bar{\Gamma}^k_{kj})\hat{\bar{\zeta}}^\beta-g_{\alpha\bar{\beta}}(\eta^k\bar{f}^\beta_{\bar{k}j}+\bar{f}^\gamma_j\bar{N}^\beta_\sigma)-(\bar{\eta}^l f^\sigma_{\bar{\mu}\bar{j}}+f^\gamma_{\bar{j}}N^\sigma_\gamma-\bar{\Gamma}^k_j f^\sigma_{\bar{k}})g_{\alpha\bar{\beta}\sigma}\hat{\bar{\zeta}}^\beta\}-\bar{\eta}^l[g^{k\bar{j}}g^{\iota\bar{\omega}}\bar{\Gamma}^h_{\bar{k}l}g_{\mu\bar{h}\bar{\upsilon}}+g^{k\bar{j}}\dot{\partial}_k(\bar{\Gamma}^h_{hl})]g_{\alpha\bar{\beta}\gamma}\hat{\bar{\zeta}}^\beta f^\gamma_{\bar{j}}-[g^{k\bar{j}}g^{\iota\bar{\omega}}\bar{\Gamma}^h_{\bar{k}\bar{j}}g_{\mu\bar{h}\bar{\upsilon}}+g^{k\bar{j}}\dot{\partial}_k(\bar{\Gamma}^h_h j)]g_{\alpha\bar{\beta}}\hat{\bar{\zeta}}^\beta.$$

注意1 若(N,L)为Hermite流形同样可以得到文献[7]中的定理3.1.

定义1 若$\frac{\partial}{\partial t}(E(f_t))\mid_{t=0}=0$,$f$称为调和的.

注意2 若(M,F)为紧Riemann曲面,可以得到文献[6]中的结果.

令

$$K(f)=E'(f)-E''(f).\tag{19}$$

则,

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