

模态耦合与能量馈入两种摩擦尖叫机理关系

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摘要: 模态耦合和能量馈入是当前重要的两个摩擦尖叫发生机理理论, 但是对于两种理论之间的关系尚未有针对性的开展研究. 针对一种典型的集总参数柔体—刚体组成的摩擦振动副, 建立 2 自由度摩擦振动动力学模型; 推导了系统产生不稳定模态耦合的条件, 并分满足模态耦合条件、满足临界模态耦合条件和不满足模态耦合条件 3 种情况分析了模态耦合理论与能量馈入理论之间的关系. 结果表明: 能量馈入依赖于模态耦合理论, 而不是一种摩擦噪声机理理论.

关键词: 摩擦尖叫; 发生机理; 模态耦合; 能量馈入

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Relationship Between Two Friction-induced Squeal Mechanisms of Mode Coupling and Energy Feed-in Theories

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Abstract: Mode coupling theory and energy feed-in theory are both regarded as important generation mechanisms of friction-induced squeal, but there have been no investigations into relationship between them so far. A two-degree-of-freedom vibration model was established to describe the dynamics of a kind of typical frictional system with friction pairs of a lumped parameter mass and a rigid mass. The conditions of mode coupling instability were derived, as well as the relationship between mode coupling and energy feed-in, when conditions of mode coupling were met, not met, or met just critical. The analysis results show that energy feed-in is dependent on mode coupling theory and cannot be regarded as generation mechanism of friction squeal.

Key words: friction-induced squeal; generation mechanism; mode coupling; energy feed-in

摩擦噪声现象在各种机械系统中广泛存在, 例如制动器、离合器和列车轮轨等^[1-2]. 它一直是机械工程学科的前沿研究热点^[1-4], 研究历史悠久, 但迄今尚未建立完全行之有效的控制手段. 究其根本原因, 就是尚未完全弄清摩擦振动和噪声的发生机理.

截至目前, 研究者提出的摩擦噪声的机理理论包括: 粘滑振动理论、模态耦合理论、自锁—滑动理论、能量馈入理论以及摩擦激励与结构耦合统一理论^[1-2, 5-27]. 但是, 国内外尚未针对各种理论开展彼此之间关系的研讨, 这无疑是非常遗憾的. 如果能够梳理出不同摩擦噪声理论间的区别或联系, 将非常有助于深化摩擦噪声发生机理, 尽快建立有效的控制手段.

在上述各种理论中, 模态耦合理论和能量馈入理论具有重要而广泛的影响. 模态耦合理论早在 1976 年由 North M R^[7] 提出, 后经 Liles G D^[28] 与有限元分析方法相结合, 已经成为学术界和工业界公认的理论之一^[1-2, 5-27]. 模态耦合理论是将摩擦力引入摩擦振动副, 使系统的刚度矩阵不对称, 产生复特征值, 由此可以根据复特征值进行稳定性判断和摩擦噪声倾向性判断的一种理论. 能量馈入理论 2001 年由管迪华教授率先提出^[29-30], 其基本思想是利用系统模态坐标下的振型向量, 计算一个振动周期内的摩擦力的能量馈入, 进而预测系统摩擦噪声倾向性, 受到国外同行的关注^[1-2, 5, 15]. 文献[31]则利用实测的摩擦力和振动加速度信号, 在物理坐标下进行了摩擦力能量馈入功率和能量的计算, 很好解释了盘—销系统摩擦噪声的时变性.

文献[29-30]针对某盘式制动器的子结构耦合模型, 计算了能量馈入量和复特征值实部, 发现二者之间具有高度相关性, 指出可以利用能量馈入量间接说明系统的不稳定性. 但遗憾的是, 文献错失了从理论上探讨能量馈入和模态耦合理论之间关系的机

会.在此背景下,本文针对一类由集总参数柔体—刚体摩擦副构成的摩擦振动系统,建立单质量 2 自由度摩擦振动动力学模型,同时采用数学公式推导和数值计算的手段,深入分析模态耦合和能量馈入这 2 种摩擦噪声理论之间的内在联系.

1 摩擦振动模型

图 1 所示为借鉴文献[27]建立的单质量 2 自由度摩擦振动动力学模型.该模型代表了一类典型的,由集总参数柔体和刚体摩擦副构成的摩擦振动副.

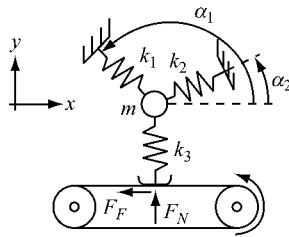


图 1 二自由度振动模型

Fig.1 2DOF vibration model

在图 1 中,集总参数柔体由具有 x 和 y 两个自由度的质点质量 m ,与 x 轴成 α_1 和 α_2 的弹簧 k_1 和 k_2 构成.柔体和刚体摩擦副之间为点接触,二者之间存在法向接触刚度 k_3 ,且以恒定的名义相对速度 v_0 ,摩擦系数为 μ .

质点质量 m 的两个自由度描述了集总参数柔体在两个正交方向上的弹性运动,对应于盘式制动器制动盘的面内模态运动与面外模态,或者盘—销系统销棒的弯曲运动与压缩运动.两个摩擦副之间存在摩擦时,会由于摩擦力作用导致摩擦副位置倾斜,因此两个角度 α_1 和 α_2 表示摩擦接触时集总参数柔体的微倾斜平衡位置,而弹簧 k_1 和 k_2 用来描述柔体在不同方向的支撑刚度.

根据力学原理,容易建立系统的动力学方程为

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -F_F \\ 0 \end{bmatrix} \quad (1)$$

式中: $k_{11} = k_1 \cos^2 \alpha_1 + k_2 \cos^2 \alpha_2$;

$$k_{12} = k_{21} = k_1 \sin \alpha_1 \cos \alpha_1 + k_2 \sin \alpha_2 \cos \alpha_2;$$

$$k_{22} = k_1 \sin^2 \alpha_1 + k_2 \sin^2 \alpha_2 + k_3;$$

$$F_F = -\mu k_3 y_1.$$

x_1, y_1 分别表示质点在 x, y 方向的位移.

定义时间尺度 $T = \sqrt{m/k_1}$, 长度尺度 $l = v_0 T$, 并令 $t_1 = Tt, x_1 = lx, y_1 = ly$; 同时, 定义刚度比 $\kappa_1 = \frac{k_2}{k_1}, \kappa_2 = \frac{k_3}{k_1}$, 以及 $k_1 = m\omega_1^2$. 若令 $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$, 则可以得

到式(1)的量纲为一的方程如下:

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{K}\mathbf{X} = \mathbf{F} \quad (2)$$

$$\text{式中, } \mathbf{M} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}; \mathbf{K} = \begin{bmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{bmatrix};$$

$$\kappa_{11} = \cos^2 \alpha_1 + \kappa_1 \cos^2 \alpha_2,$$

$$\kappa_{12} = \kappa_{21} = \sin \alpha_1 \cos \alpha_1 + \kappa_1 \sin \alpha_2 \cos \alpha_2,$$

$$\kappa_{22} = \sin^2 \alpha_1 + \kappa_1 \sin^2 \alpha_2 + \kappa_2;$$

$$\mathbf{F} = \begin{bmatrix} -f \\ 0 \end{bmatrix}, f = -\mu \kappa_2 y.$$

把方程进一步化为

$$\ddot{\mathbf{X}} + \mathbf{A}\mathbf{X} = \mathbf{F} \quad (3)$$

$$\text{式中, } \mathbf{A} = \begin{bmatrix} \kappa_{11} & \kappa_{12} - \mu \kappa_2 \\ \kappa_{21} & \kappa_{22} \end{bmatrix}, \mathbf{F} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

2 模态耦合不稳定性条件

为了得到系统(3)的特征方程,令 $\mathbf{X} = \mathbf{V}\Omega t$, 其中 \mathbf{V} 为特征向量,由此可得关于 Ω 的特征方程为

$$\begin{vmatrix} \kappa_{11} + \Omega^2 & \kappa_{12} - \mu \kappa_2 \\ \kappa_{21} & \kappa_{22} + \Omega^2 \end{vmatrix} = 0 \quad (4)$$

可求得特征值为

$$\Omega_{1,2}^2 = -\frac{1}{2}(1 + \kappa_1 + \kappa_2) \pm \frac{1}{2}\sqrt{\Delta} \quad (5)$$

式中, $\Delta = 1 + \kappa_1^2 + \kappa_2^2 + 2\kappa_1 \cos(2\alpha_1 - 2\alpha_2) - 2\kappa_2 \cos 2\alpha_1 - 2\kappa_1 \kappa_2 \cos 2\alpha_2 - 2\mu[\kappa_2 \sin 2\alpha_1 + \kappa_1 \kappa_2 \sin 2\alpha_2]$.

根据 Ω_1 和 Ω_2 的不同情况进行系统稳定性讨论:

(1) 当 $\Delta > 0$ 时, Ω_1 和 Ω_2 都是一对共轭的纯虚数,但不相同,此时系统稳定.

(2) 当 $\Delta = 0$ 时, Ω_1 和 Ω_2 都是一对共轭的纯虚数,且相同,此时系统不稳定.

(3) 当 $\Delta < 0$ 时, Ω_1 和 Ω_2 都是一对共轭的复数,此时系统不稳定,系统有一个稳定模态和一个不稳定模态,产生模态耦合不稳定现象.

综上所述,当系统的结构参数和摩擦接触参数满足式(6)时系统将可能产生模态耦合,导致不稳定. $\Delta < 0$

$$\begin{aligned} & \text{由于: } 1 + \kappa_1^2 + \kappa_2^2 + 2\kappa_1 \cos(2\alpha_1 - 2\alpha_2) \\ & - 2\kappa_2 \cos 2\alpha_1 - 2\kappa_1 \kappa_2 \cos 2\alpha_2 \geq 0, \text{ 因此, 当且仅当 } \sin(2\alpha_1) + \kappa_1 \sin(2\alpha_2) > 0, \text{ 且 } \mu > \\ & 1 + \kappa_1^2 + \kappa_2^2 + 2\kappa_1 \cos(2\alpha_1 - 2\alpha_2) \\ & - \frac{2\kappa_2 \cos 2\alpha_1 - 2\kappa_1 \kappa_2 \cos 2\alpha_2}{2\kappa_2 \sin 2\alpha_1 + 2\kappa_1 \kappa_2 \sin 2\alpha_2} \text{ 时, 系统产生模态耦合.} \end{aligned} \quad (6)$$

取系统参数(1): $m=1 \text{ kg}$, $\alpha_1=150^\circ$, $\alpha_2=30^\circ$, $k_1=\frac{0.2}{3}(2-\sqrt{3})\text{N}\cdot\text{m}^{-1}$, $\kappa_1=\frac{2+\sqrt{3}}{2-\sqrt{3}}$, $\kappa_2=\frac{2}{2-\sqrt{3}}$, 则系统模态耦合与摩擦系数的关系如图 2 所示. 由图 2 可知: 当摩擦系数 $\mu>0.75$ 时, 系统发生模态耦合.

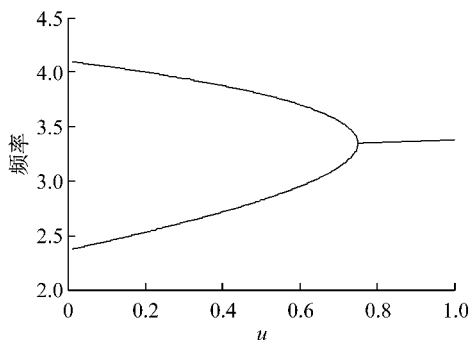


图 2 模态耦合与摩擦系数的关系

Fig.2 The relationship between mode coupling and friction coefficient

3 模态耦合不稳定与能量馈入的关系

假设 x, y, \dot{x} 和 \dot{y} 的初值任意, 令 $\mathbf{Y} = [x \ y \ \dot{x} \ \dot{y}]^T$, 式(3)转化为状态空间方程

$$\dot{\mathbf{Y}} = \mathbf{B}\mathbf{Y} \quad (7)$$

$$\text{式中: } \mathbf{B} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\kappa_{11} & -(\kappa_{12} - \mu\kappa_2) & 0 & 0 \\ -\kappa_{21} & -\kappa_{22} & 0 & 0 \end{bmatrix}.$$

为了得到系统(7)的特征方程, 令 $\mathbf{Y} = \mathbf{U}\Omega t$, 其中 \mathbf{U} 为特征向量, 则可以建立关于 Ω 的特征方程为

$$|\Omega \mathbf{E} - \mathbf{B}| = \begin{vmatrix} \Omega & 0 & -1 & 0 \\ 0 & \Omega & 0 & -1 \\ \kappa_{11} & \kappa_{12} - \mu\kappa_2 & \Omega & 0 \\ \kappa_{21} & \kappa_{22} & 0 & \Omega \end{vmatrix} = 0 \quad (8)$$

化简为

$$\begin{vmatrix} \kappa_{11} + \Omega^2 & \kappa_{12} - \mu\kappa_2 \\ \kappa_{21} & \kappa_{22} + \Omega^2 \end{vmatrix} = 0 \quad (9)$$

由于式(9)和式(4)相同, 由此也可以得到特征值表达式(5)以及模态耦合条件.

下面, 将分 3 种情况, 讨论模态耦合与能量馈入之间的关系.

3.1 满足模态耦合条件时

不妨假设 $\Omega = \pm(a \pm bi)$, 其中 $a>0, b>0$, 则

$$\begin{cases} a^2 - b^2 = -\frac{1}{2}(1 + \kappa_1 + \kappa_2) \\ 2ab = \frac{1}{2}\sqrt{-\Delta} \end{cases} \quad (10)$$

解得

$$\mathbf{Y} = \begin{bmatrix} \kappa_{12} - \mu\kappa_2 & 0 \\ -(\kappa_{11} + a^2 - b^2) & -2ab \\ a(\kappa_{12} - \mu\kappa_2) & b(\kappa_{12} - \mu\kappa_2) \\ a(-\kappa_{11} - a^2 + 3b^2) & b(-\kappa_{11} - 3a^2 + b^2) \\ \kappa_{12} - \mu\kappa_2 & 0 \\ -(\kappa_{11} + a^2 - b^2) & 2ab \\ -a(\kappa_{12} - \mu\kappa_2) & b(\kappa_{12} - \mu\kappa_2) \\ -a(-\kappa_{11} - a^2 + 3b^2) & b(-\kappa_{11} - 3a^2 + b^2) \end{bmatrix} \cdot \begin{bmatrix} c_1 e^{at} \cos bt + c_2 e^{at} \sin bt \\ -c_1 e^{at} \sin bt + c_2 e^{at} \cos bt \\ c_3 e^{-at} \cos bt + c_4 e^{-at} \sin bt \\ -c_3 e^{-at} \sin bt + c_4 e^{-at} \cos bt \end{bmatrix} \quad (11)$$

式中: c_1, c_2, c_3 和 c_4 为任意常数, 且由初始条件确定.

由式(11)可得

$$x(t) = C_1 e^{at} \sin(bt + \phi_1) + C_2 e^{-at} \sin(bt + \phi_2) \quad (12)$$

$$\begin{aligned} \text{式中: } C_1 &= (\kappa_{12} - \mu\kappa_2) \sqrt{c_1^2 + c_2^2}, C_2 = (\kappa_{12} - \mu\kappa_2) \\ &\sqrt{c_3^2 + c_4^2}, \sin \phi_1 = \frac{c_1}{\sqrt{c_1^2 + c_2^2}}, \cos \phi_1 = \frac{c_2}{\sqrt{c_1^2 + c_2^2}}, \sin \phi_2 \\ &= \frac{c_3}{\sqrt{c_3^2 + c_4^2}}, \cos \phi_2 = \frac{c_4}{\sqrt{c_3^2 + c_4^2}}. \end{aligned}$$

$$y(t) = C[C_1 e^{at} \sin(bt + \phi_1 - \phi_3) + C_2 e^{-at} \sin(bt + \phi_2 + \phi_3)] \quad (13)$$

$$\text{式中: } C = \frac{\sqrt{(\kappa_{11} + a^2 - b^2)^2 + (2ab)^2}}{\kappa_{12} - \mu\kappa_2},$$

$$\sin \phi_3 = \frac{2ab}{\sqrt{(\kappa_{11} + a^2 - b^2)^2 + (2ab)^2}},$$

$$\cos \phi_3 = \frac{-(\kappa_{11} + a^2 - b^2)}{\sqrt{(\kappa_{11} + a^2 - b^2)^2 + (2ab)^2}}.$$

显然, 当存在模态耦合时, x 与 y 方向振动的相位差取决于系统的摩擦副结构参数和摩擦接触参数.

考虑到仅摩擦力是非保守力, 故摩擦系统向摩擦副的馈入能量可用一个周期内系统总能量的变化来表示.

不妨假设振动周期为 $\tau_1 = \frac{2\pi}{b}$, 则摩擦副的动能

为

$$T(t) = \frac{1}{2} [\dot{x} \quad \dot{y}] \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \frac{1}{2} (\dot{x}^2 + \dot{y}^2) \quad (14)$$

摩擦副的势能为

$$U(t) = \frac{1}{2} [x \quad y] \cdot \begin{bmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} (\kappa_{11}x^2 + \kappa_{22}y^2 + 2\kappa_{12}xy) \quad (15)$$

摩擦副的总能量为

$$E(t) = T(t) + U(t) = \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + \kappa_{11}x^2 + \kappa_{22}y^2 + 2\kappa_{12}xy) \quad (16)$$

一个振动周期内的动能、势能和总能量变化为

$$\begin{aligned} \Delta T(t) &= T(t + \tau_1) - T(t) = \frac{1}{2} C_1^2 (e^{2a(t+\tau_1)} - e^{2at}) [(a \sin(bt + \phi_1) + b \cos(bt + \phi_1))^2 + \\ &C^2 (a \sin(bt + \phi_1 - \phi_3) + b \cos(bt + \phi_1 - \phi_3))^2] + \frac{1}{2} C_2^2 (e^{-2a(t+\tau_1)} - e^{-2at}) [(-a \sin(bt + \phi_2) + b \cos(bt + \phi_2))^2 + \\ &C^2 (-a \sin(bt + \phi_2 + \phi_3) + b \cos(bt + \phi_2 + \phi_3))^2] \quad (17) \end{aligned}$$

$$\begin{aligned} \Delta U(t) &= U(t + \tau_1) - U(t) = \frac{1}{2} C_1^2 (e^{2a(t+\tau_1)} - e^{2at}) [(\cos \alpha_1 \sin(bt + \phi_1) + C \sin \alpha_1 \sin(bt + \phi_1 - \phi_3))^2 + \kappa_1 (\cos \alpha_2 \sin(bt + \phi_1) + \\ &C \sin \alpha_2 \sin(bt + \phi_1 - \phi_3))^2 + \kappa_2 C^2 \sin^2(bt + \phi_1 - \phi_3)] + \frac{1}{2} C_2^2 (e^{-2a(t+\tau_1)} - e^{-2at}) [(\cos \alpha_1 \sin(bt + \phi_2) + C \sin \alpha_1 \sin(bt + \phi_2 + \phi_3))^2 + \kappa_1 (\cos \alpha_2 \sin(bt + \phi_2) + \\ &C \sin \alpha_2 \sin(bt + \phi_2 + \phi_3))^2 + \kappa_2 C^2 \sin^2(bt + \phi_2 + \phi_3)] \quad (18) \end{aligned}$$

$$\begin{aligned} \Delta E(t) &= \Delta T(t) + \Delta U(t) = \frac{1}{2} C_1^2 (e^{2a(t+\tau_1)} - e^{2at}) [(a \sin(bt + \phi_1) + b \cos(bt + \phi_1))^2 + \\ &C^2 (a \sin(bt + \phi_1 - \phi_3) + b \cos(bt + \phi_1 - \phi_3))^2 + (\cos \alpha_1 \sin(bt + \phi_1) + C \sin \alpha_1 \sin(bt + \phi_1 - \phi_3))^2 + \kappa_1 (\cos \alpha_2 \sin(bt + \phi_1) + \\ &C \sin \alpha_2 \sin(bt + \phi_1 - \phi_3))^2 + \kappa_2 C^2 \sin^2(bt + \phi_1 - \phi_3)] + \frac{1}{2} C_2^2 (e^{-2a(t+\tau_1)} - e^{-2at}) [(-a \sin(bt + \phi_2) + b \cos(bt + \phi_2))^2 + \\ &C^2 (-a \sin(bt + \phi_2 + \phi_3) + b \cos(bt + \phi_2 + \phi_3))^2 + (\cos \alpha_1 \sin(bt + \phi_2) + C \sin \alpha_1 \sin(bt + \phi_2 + \phi_3))^2 + \kappa_1 (\cos \alpha_2 \sin(bt + \phi_2) + \\ &C \sin \alpha_2 \sin(bt + \phi_2 + \phi_3))^2 + \kappa_2 C^2 \sin^2(bt + \phi_2 + \phi_3)] \end{aligned}$$

$$\phi_2 + \phi_3)] \quad (19)$$

$$\begin{aligned} p(t) &= \frac{1}{2} C_1^2 [(a \sin(bt + \phi_1) + b \cos(bt + \phi_1))^2 + C^2 (a \sin(bt + \phi_1 - \phi_3) + b \cos(bt + \phi_1 - \phi_3))^2 + (\cos \alpha_1 \sin(bt + \phi_1) + C \sin \alpha_1 \sin(bt + \phi_1 - \phi_3))^2 + \kappa_1 (\cos \alpha_2 \sin(bt + \phi_1) + C \sin \alpha_2 \sin(bt + \phi_1 - \phi_3))^2 + \kappa_2 C^2 \sin^2(bt + \phi_1 - \phi_3)] \quad (20) \end{aligned}$$

$$\begin{aligned} q(t) &= \frac{1}{2} C_2^2 [(-a \sin(bt + \phi_2) + b \cos(bt + \phi_2))^2 + C^2 (-a \sin(bt + \phi_2 + \phi_3) + b \cos(bt + \phi_2 + \phi_3))^2 + (\cos \alpha_1 \sin(bt + \phi_2) + C \sin \alpha_1 \sin(bt + \phi_2 + \phi_3))^2 + \kappa_1 (\cos \alpha_2 \sin(bt + \phi_2) + C \sin \alpha_2 \sin(bt + \phi_2 + \phi_3))^2 + \kappa_2 C^2 \sin^2(bt + \phi_2 + \phi_3)] \quad (21) \end{aligned}$$

显然,

$$\begin{cases} p(t) = p(t + \tau_1) > 0 \\ q(t) = q(t + \tau_1) > 0 \end{cases} \quad (22)$$

则

$$\Delta E(t) = (p(t + \tau_1)e^{2a(t+\tau_1)} + q(t + \tau_1)e^{-2a(t+\tau_1)}) - (p(t)e^{2at} + q(t)e^{-2at}) \quad (23)$$

根据函数 $p(t)e^{2at} + q(t)e^{-2at}$ 的性质,一定时间后,可认为 $p(t)e^{2at} + q(t)e^{-2at} \approx p(t)e^{2at}$,因此经过一定的时间后,存在 $\Delta E(t) = p(t)(e^{2a(t+\tau_1)} - e^{2at}) > 0$. 这说明,当满足模态耦合条件时,经过一定的时间后,一个周期内摩擦力对振动副所做的正功大于负功,这就造成摩擦系统向振动副馈入的能量总是大于振动副向摩擦系统馈出的能量,因此产生了振动副的能量馈入现象,系统失稳并产生摩擦噪声.

取系统参数(I)以及 $\mu=0.8$,根据图 2 可知此时系统存在模态耦合. 设初值 $x=y=0, \dot{x}=\dot{y}=0.5$,计算得到模态耦合条件下系统动能、势能和总能量的变化时间历程,如图 3 所示. 显然,一个周期内,振动

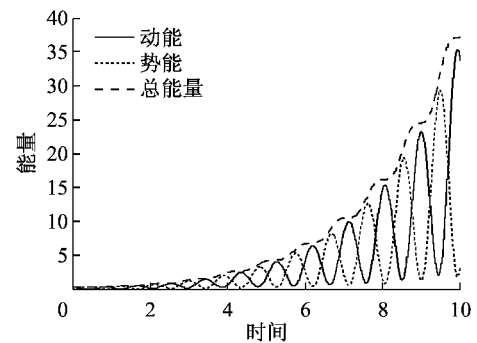


图 3 满足模态耦合条件时的振动副能量变化

Fig. 3 The energy change of vibration system when conditions of mode coupling are met

副馈入的能量总是大于馈出的能量,振动副能量不断增加,与理论分析相符.

3.2 不满足模态耦合条件时

假设 $\Omega_1 = \pm a_1 i, \Omega_2 = \pm b_1 i$, 其中 $a_1 > b_1 > 0$, 则

$$a_1^2 = \frac{1}{2}(1 + \kappa_1 + \kappa_2) + \frac{1}{2}\sqrt{\Delta} \quad (24)$$

$$\Phi = \begin{bmatrix} \kappa_{12} - \mu\kappa_2 & 0 & \kappa_{12} - \mu\kappa_2 & 0 \\ -\kappa_{11} + a_1^2 & 0 & -\kappa_{11} + b_1^2 & 0 \\ 0 & a_1(\kappa_{12} - \mu\kappa_2) & 0 & b_1(\kappa_{12} - \mu\kappa_2) \\ 0 & a_1(-\kappa_{11} + a_1^2) & 0 & b_1(-\kappa_{11} + b_1^2) \end{bmatrix}$$

$$\text{使得 } \Phi^{-1}B\Phi = \begin{bmatrix} 0 & a_1 & 0 & 0 \\ -a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_1 \\ 0 & 0 & -b_1 & 0 \end{bmatrix} \quad (26)$$

定义 $Z = (z_1, z_2, z_3, z_4)^T$, 求解方程 $\dot{Z} = \Phi^{-1}B\Phi Z$, 可得

$$Z = \begin{bmatrix} c_5 \cos a_1 t + c_6 \sin a_1 t \\ -c_5 \sin a_1 t + c_6 \cos a_1 t \\ c_7 \cos b_1 t + c_8 \sin b_1 t \\ -c_7 \sin b_1 t + c_8 \cos b_1 t \end{bmatrix} \quad (27)$$

式中, c_5, c_6, c_7, c_8 为任意常数, 且由初始条件决定.

因为 $Y = \Phi Z$, 则

$$x(t) = C_3 \sin(a_1 t + \beta_1) + C_4 \sin(b_1 t + \beta_2) \quad (28)$$

$$\text{式中: } C_3 = (\kappa_{12} - \mu\kappa_2) \sqrt{c_5^2 + c_6^2}, C_4 = (\kappa_{12} - \mu\kappa_2) \sqrt{c_7^2 + c_8^2}, \sin \beta_1 = \frac{c_5}{\sqrt{c_5^2 + c_6^2}}, \cos \beta_1 = \frac{c_6}{\sqrt{c_5^2 + c_6^2}},$$

$$\sin \beta_2 = \frac{c_7}{\sqrt{c_7^2 + c_8^2}}, \cos \beta_2 = \frac{c_8}{\sqrt{c_7^2 + c_8^2}}.$$

$$y(t) = C_5 \sin(a_1 t + \beta_1) + C_6 \sin(b_1 t + \beta_2) \quad (29)$$

$$\text{式中: } C_5 = (-\kappa_{11} + a_1^2) \sqrt{c_5^2 + c_6^2}, C_6 = (-\kappa_{11} + b_1^2) \sqrt{c_7^2 + c_8^2}.$$

若 $\frac{a_1}{b_1}$ 是有理数, 则必存在互质自然数 n 和 m , 使

得 $\frac{a_1}{b_1} = \frac{n}{m}$. 因此, 振动周期 $\tau_2 = \frac{2\pi}{a_1}n = \frac{2\pi}{b_1}m$, 为原来各自周期的整数倍, 运动为周期运动. 在这种情况下, $\Delta E(t) = 0$, 即在一个周期内, 由于摩擦力做功, 导致存在摩擦系统与振动副能量的相互传递, 但由于摩擦力对振动副所做的正功等于负功, 导致经过整数个周期后的振动副总能量保持不变, 即一个周期内, 振动副馈入的能量等于馈出的能量.

若 $\frac{a_1}{b_1}$ 是无理数, 可在极坐标下考虑此问题. 令 r_1^2

$$b_1^2 = \frac{1}{2}(1 + \kappa_1 + \kappa_2) - \frac{1}{2}\sqrt{\Delta} \quad (25)$$

显然, 存在可逆矩阵

$$= z_1^2 + z_2^2, r_2^2 = z_3^2 + z_4^2, \tan \theta_1 = \frac{z_2}{z_1}, \tan \theta_2 = \frac{z_4}{z_3}, \text{ 则可得}$$

得 $\dot{r}_1 = 0, \dot{\theta}_1 = -a_1, \dot{r}_2 = 0, \dot{\theta}_2 = -b_1$. 可将 θ_1 和 θ_2 看成是边长为 2π 的正方形中的变量 (只要将正方形的对边 $\theta_i = 0$ 和 $\theta_i = 2\pi$ 粘合起来就可得到环面). 在正方形中, 向量场的斜率为常数 $\dot{\theta}_2 / \dot{\theta}_1 = b_1 / a_1$, 从而可定义庞加莱映射 $g(x_0) = (x_0 + 2\pi \frac{b_1}{a_1}) \bmod 2\pi$, 其中 $x_0 = \theta_2(0)$, 点集 $x_0, x_1 = g(x_0), x_2 = g(x_1), \dots, x_n = g(x_{n-1})$ 构成 x_0 在 g 迭代下的轨道. 可证明, 总存在整数 $m > n$, 对任意的 $\xi > 0, |x_{k(m-n)} - x_{(k-1)(m-n)}| < \xi, k = 1, 2, 3, \dots$, 即通过一次次旋转角度 $2\pi(m-n)\frac{b_1}{a_1}$, 可得到一系列点, 其中相邻的两点之间的距离都在 ξ 内, 因而 Z 在其所在的环境上是稠密的, $Y = \Phi Z$ 不改变其稠密性, 这种形式的运动为准周期运动. 因此, 经过某个旋转周期, 振动副总能量与一个旋转周期前可任意接近, 可以认为任意时刻经过整数个周期后的总能量几乎保持不变. 而在此旋转周期内, 摩擦系统与振动副的能量相互传递, 即也可认为一个周期内, 振动副馈入的能量等于馈出的能量.

取系统参数(I)以及 $\mu = 0.6$, 根据图 2 可知系统不满足模态耦合条件. 将初值设定为 $x = y = 0, \dot{x} = \dot{y} = 0.5$, 此时 $\frac{a_1}{b_1}$ 是无理数. 由图 4 可知, 此时 x 和 y 在

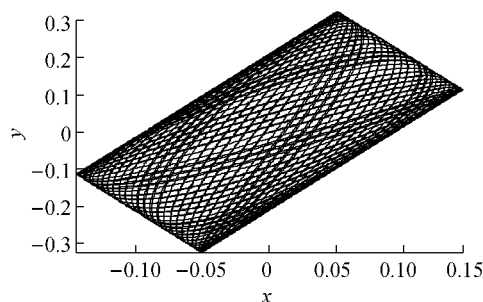


图4 x, y 在平面上的投影

Fig. 4 The projection of x, y in the plane

四边形面内稠密,若增加仿真时间,则解布满整个四边形面.图5为该情况下的能量变化情况仿真结果,振动副能量在一个周期前后保持不变,振动副馈入的能量等于馈出的能量,与理论分析相符.

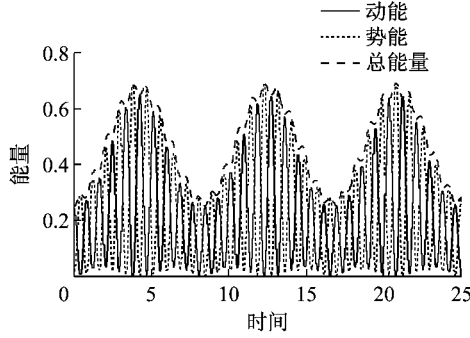


图5 不满足模态耦合条件时的振动副能量变化

Fig.5 The energy change of vibration system when conditions of mode coupling are not met

3.3 在模态耦合临界条件下

假设 $\Omega_1 = \Omega_2 = \pm a_2 i$, 其中 $a_2 > 0$, 则

$$a_2^2 = \frac{1}{2}(1 + \kappa_1 + \kappa_2) \quad (30)$$

矩阵 B 的等价标准型为

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & (\Omega^2 + \kappa_{11})(\Omega^2 + \kappa_{22}) - \kappa_{21}(\kappa_{12} - \mu\kappa_2) \end{bmatrix},$$

矩阵 B 的最小多项式为 $(\Omega^2 + \kappa_{11})(\Omega^2 + \kappa_{22}) - \kappa_{21}(\kappa_{12} - \mu\kappa_2)$, 显然有重根.

求解可得

$$Y = \begin{bmatrix} \kappa_{12} - \mu\kappa_2 & 0 & 0 & 0 \\ -\kappa_{11} + a_2^2 & 0 & 0 & -2a_2 \\ 0 & a_2(\kappa_{12} - \mu\kappa_2) & \kappa_{12} - \mu\kappa_2 & 0 \\ 0 & -a_2(\kappa_{11} - a_2^2) & \kappa_{22} + a_2^2 & 0 \end{bmatrix} \cdot \begin{bmatrix} D_1 t \cos a_2 t + D_2 t \sin a_2 t + D_3 \cos a_2 t + D_4 \sin a_2 t \\ -D_1 t \sin a_2 t + D_2 t \cos a_2 t - D_3 \sin a_2 t + D_4 \cos a_2 t \\ D_1 \cos a_2 t + D_2 \sin a_2 t \\ -D_1 \sin a_2 t + D_2 \cos a_2 t \end{bmatrix} \quad (31)$$

式中, D_1, D_2, D_3, D_4 为任意常数.

令 $d_1 = \sqrt{D_1^2 + D_2^2}, d_2 = \sqrt{D_3^2 + D_4^2}$,

$$\begin{cases} \sin \delta_1 = \frac{D_1}{\sqrt{D_1^2 + D_2^2}} \\ \cos \delta_1 = \frac{D_2}{\sqrt{D_1^2 + D_2^2}} \end{cases}, \begin{cases} \sin \delta_2 = \frac{D_3}{\sqrt{D_3^2 + D_4^2}} \\ \cos \delta_2 = \frac{D_4}{\sqrt{D_3^2 + D_4^2}} \end{cases}, \text{ 则}$$

$$\begin{cases} x = (\kappa_{12} - \mu\kappa_2)[d_1 t \sin(a_2 t + \delta_1) + d_2 \sin(a_2 t + \delta_2)] \\ y = (-\kappa_{11} + a_2^2)[d_1 t \cos(a_2 t + \delta_1) + d_2 \cos(a_2 t + \delta_2)] - 2a_2 d_1 \cos(a_2 t + \delta_1) \end{cases} \quad (32)$$

不妨假设振动周期为 $\tau_3 = \frac{2\pi}{a_2}$, 则系统的动能变化量为

$$\begin{aligned} \Delta T = T(t + \tau_3) - T(t) &= \frac{1}{2}(\kappa_{12} - \mu\kappa_2)^2 a_2^2 d_1^2 \tau_3 (2t + \tau_3) \cos^2(a_2 t + \delta_1) + \frac{1}{2}(-\kappa_{11} + a_2^2)^2 a_2^2 d_1^2 \tau_3 (2t + \tau_3) \sin^2(a_2 t + \delta_1) \\ &+ (\kappa_{12} - \mu\kappa_2)^2 a_2^2 d_1 d_2 \tau_3 \sin(a_2 t + \delta_1) \cos(a_2 t + \delta_1) + (\kappa_{12} - \mu\kappa_2)^2 a_2^2 d_1 d_2 \tau_3 \cos(a_2 t + \delta_2) \cos(a_2 t + \delta_1) \\ &- a_2(-\kappa_{11} + a_2^2)^2 d_1^2 \tau_3 \sin(a_2 t + \delta_1) \cos(a_2 t + \delta_1) + a_2^2(-\kappa_{11} + a_2^2)^2 d_1 d_2 \tau_3 \sin(a_2 t + \delta_1) \sin(a_2 t + \delta_2) \\ &- 2(-\kappa_{11} + a_2^2) a_2^2 d_1^2 \tau_3 \sin^2(a_2 t + \delta_1) \end{aligned} \quad (33)$$

定义: $A(t) = \frac{1}{2}(\kappa_{12} - \mu\kappa_2)^2 a_2^2 d_1^2 \tau_3 (2t + \tau_3) \cos^2(a_2 t + \delta_1) + \frac{1}{2}(-\kappa_{11} + a_2^2)^2 a_2^2 d_1^2 \tau_3 (2t + \tau_3) \sin^2(a_2 t + \delta_1)$,

显然 $\lim_{t \rightarrow +\infty} A(t) = +\infty$.

定义:

$$\begin{aligned} B(t) &= (\kappa_{12} - \mu\kappa_2)^2 a_2^2 d_1^2 \tau_3 \sin(a_2 t + \delta_1) \cos(a_2 t + \delta_1) + (\kappa_{12} - \mu\kappa_2)^2 a_2^2 d_1 d_2 \tau_3 \cos(a_2 t + \delta_2) \cos(a_2 t + \delta_1) \\ &- a_2(-\kappa_{11} + a_2^2)^2 d_1^2 \tau_3 \sin(a_2 t + \delta_1) \cos(a_2 t + \delta_1) + a_2^2(-\kappa_{11} + a_2^2)^2 d_1 d_2 \tau_3 \sin(a_2 t + \delta_1) \sin(a_2 t + \delta_2) \\ &- 2(-\kappa_{11} + a_2^2) a_2^2 d_1^2 \tau_3 \sin^2(a_2 t + \delta_1), \end{aligned}$$

显然 $B(t)$ 有界. 因此, 必定存在某时刻 t_1 , 当 $t > t_1$ 时, 有 $\Delta T = A(t) + B(t) > 0$.

系统的势能变化量为

$$\begin{aligned} \Delta U = U(t + \tau_3) - U(t) &= (t \cdot \tau_3 + \frac{1}{2} \tau_3^2) d_1^2 [\cos a_1 (\kappa_{12} - \mu\kappa_2) \sin(a_2 t + \delta_1) + \sin a_1 (-\kappa_{11} + a_2^2) \cos(a_2 t + \delta_1)]^2 \\ &+ \kappa_{11} (t \cdot \tau_3 + \frac{1}{2} \tau_3^2) d_1^2 [\cos a_2 (\kappa_{12} - \mu\kappa_2) \sin(a_2 t + \delta_1) + \sin a_2 (-\kappa_{11} + a_2^2) \cos(a_2 t + \delta_1)]^2 \\ &+ \kappa_2 (-\kappa_{11} + a_2^2)^2 (t \cdot \tau_3 + \frac{1}{2} \tau_3^2) d_1^2 \cos^2(a_2 t + \delta_1) + \kappa_{11} (\kappa_{12} - \mu\kappa_2)^2 d_1 d_2 \tau_3 \sin(a_2 t + \delta_1) \sin(a_2 t + \delta_2) \\ &+ \kappa_{22} (-\kappa_{11} + a_2^2)^2 d_1 d_2 \tau_3 \cos(a_2 t + \delta_1) \cos(a_2 t + \delta_2) \end{aligned}$$

$$\begin{aligned} & \delta_2) - 2\kappa_{22}a_2d_1^2\tau_3\cos^2(a_2t+\delta_1)(-\kappa_{11}+a_2^2) + \\ & \kappa_{12}(-\kappa_{11}+a_2^2)(\kappa_{12}-\mu\kappa_2)d_1d_2\tau_3\sin(2a_2t+ \\ & \delta_1+\delta_2) - 2\kappa_{12}a_2d_1^2(\kappa_{12}-\mu\kappa_2)\tau_3\sin(a_2t+ \\ & \delta_1)\cos(a_2t+\delta_1) + \kappa_{12}(-\kappa_{11}+a_2^2)(\kappa_{12}- \\ & \mu\kappa_2)d_1^2\tau_3^2\sin(a_2t+\delta_1)\cos(a_2t+\delta_1) \quad (34) \end{aligned}$$

定义:

$$\begin{aligned} C(t) = & (t \cdot \tau_3 + \frac{1}{2}\tau_3^2)d_1^2[\cos\alpha_1(\kappa_{12}-\mu\kappa_2)\sin(a_2t+ \\ & \delta_1) + \sin\alpha_1(-\kappa_{11}+a_2^2)\cos(a_2t+\delta_1)]^2 + \\ & \kappa_1(t \cdot \tau_3 + \frac{1}{2}\tau_3^2)d_1^2[\cos\alpha_2(\kappa_{12}-\mu\kappa_2)\sin(a_2t+ \\ & \delta_1) + \sin\alpha_2(-\kappa_{11}+a_2^2)\cos(a_2t+\delta_1)]^2 + \\ & \kappa_2(-\kappa_{11}+a_2^2)^2(t \cdot \tau_3 + \frac{1}{2}\tau_3^2)d_1^2\cos^2(a_2t+ \\ & \delta_1), \end{aligned}$$

显然, $\lim_{t \rightarrow +\infty} C(t) = +\infty$.

定义

$$\begin{aligned} D(t) = & \kappa_{11}(\kappa_{12}-\mu\kappa_2)^2d_1d_2\tau_3\sin(a_2t+\delta_1)\sin(a_2t+ \\ & \delta_2) + \kappa_{22}(-\kappa_{11}+a_2^2)^2d_1d_2\tau_3\cos(a_2t+ \\ & \delta_1)\cos(a_2t+\delta_2) - 2\kappa_{22}a_2d_1^2\tau_3\cos^2(a_2t+ \\ & \delta_1)(-\kappa_{11}+a_2^2) + \kappa_{12}(-\kappa_{11}+a_2^2)(\kappa_{12}- \\ & \mu\kappa_2)d_1d_2\tau_3\sin(2a_2t+\delta_1+\delta_2) - \\ & 2\kappa_{12}a_2d_1^2(\kappa_{12}-\mu\kappa_2)\tau_3\sin(a_2t+\delta_1)\cos(a_2t+ \\ & \delta_1) + \kappa_{12}(-\kappa_{11}+a_2^2)(\kappa_{12}-\mu\kappa_2)d_1^2\tau_3^2\sin(a_2t+ \\ & \delta_1)\cos(a_2t+\delta_1), \end{aligned}$$

显然, $D(t)$ 有界.

因此, 必定存在某时刻 t_2 , 当 $t > t_2$ 时, 有 $\Delta T = C(t) + D(t) > 0$.

综上, 若取 $t_0 = \max(t_1, t_2)$, 当 $t > t_0$ 时, 有 $\Delta E = \Delta U + \Delta T > 0$, 即一定时间后, 一个周期内, 摩擦系统馈入的能量将总是大于馈出的能量, 振动副能量不断增大.

取系统参数(I)以及 $\mu = 0.75$, 根据图 2 可知, 系统处于临界模态耦合状态. 将初值设定为 $x = y = 0$, $\dot{x} = \dot{y} = 0.5$, 图 6 为该情况下的能量仿真结果, 显然一个周期内, 振动副馈入的能量总是大于馈出的能量, 振动副能量不断增加, 与理论分析相符. 值得注意的是, 临界条件事实上又分两种情况: ① B 的最小多项式有重根, 此时系统是不稳定的, 前面分析的就属于这种情况; ② B 的最小多项式无重根, 此时临界条件与不满足模态耦合且两频率之比为有理数的情况类似.

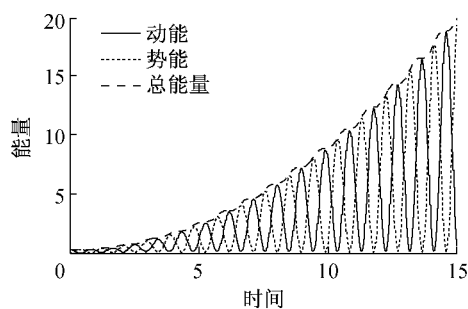


图 6 模态耦合临界条件下的振动副能量变化

Fig. 6 The energy change of vibration system when conditions of mode coupling are met just critical

4 结论

(1) 满足模态耦合条件时, 一定时间后, 一个周期内摩擦系统向振动副馈入的能量必定大于振动副馈出的能量, 振动副能量不断增大, 且这与初始条件无关; 在模态耦合临界条件下, 能量馈入情况与模态耦合时类似.

(2) 当不满足模态耦合条件时, 任意周期后振动副能量无明显变化; 而在一个周期内, 摩擦系统与振动副的能量相互传递, 即摩擦系统向振动副馈入的能量等于振动副馈出的能量.

因此, 摩擦振动副是否发生能量馈入依赖于系统是否满足模态耦合条件. 这不仅与文献[29-30]通过数值计算发现能量馈入量与特征值实部高度相关, 且可以用于判断摩擦噪声倾向性的结论一致, 而且也与文献[31]中的能量馈入量可以解释摩擦噪声幅值的变化原因的结论相一致. 综上所述, 能量馈入依赖于模态耦合理论, 能量馈入不是一种摩擦噪声发生机理.

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