第 45 卷第 6 期
2017 年 6 月

同 济 大 学 学 报(自 然 科 学 版)
JOURNAL OF TONGJI UNIVERSITY(NATURAL SCIENCE)

Vol. 45 No. 6
Jun. 2017

文章编号： 0253-374X(2017)06-0930-06

DOI：10.11908/j.issn.0253-374x.2017.06.022

$\mathrm{End}_{s_k(2,5)}(\Omega_k^{\otimes 5})$ 与 $\mathrm{End}_{u_k(2,5)}(\Omega_k^{\otimes 5})$ 的结构

卞之豪

(同济大学 数学科学学院，上海 200092)

摘要： 利用张量空间 $\Omega_k^{\otimes 5}$ 作为 $\mathrm{gl}_k(2)$ 量子包络代数(q-Schur 代数)的 tilting 模分解以及在 $n=2$ 时已知的 tilting 模结构，给出 $\Omega_k^{\otimes 5}$ 作为无穷小 q-Schur 代数及小 q-Schur 代数的模时 $\mathrm{End}_{s_k(2,5)}(\Omega_k^{\otimes 5})$ 与 $\mathrm{End}_{u_k(2,5)}(\Omega_k^{\otimes 5})$ 的维数、一组生成元以及主模分解.

关键词： 无穷小 q-Schur 代数；小 q-Schur 代数；Schur-Weyl 对偶

中图分类号： O152.5　　**文献标志码：** A

On the Structure of $\mathrm{End}_{s_k(2,5)}(\Omega_k^{\otimes 5})$ and $\mathrm{End}_{u_k(2,5)}(\Omega_k^{\otimes 5})$

BIAN Zhihao

(School of Mathematical Sciences, Tongji University, Shanghai 200092, China)

Abstract: By using the fact that tensor space $\Omega_k^{\otimes 5}$ can be written as a direct sum of tilting modules for the quantum enveloping algebra (q-Schur algebra) of $\mathrm{gl}_k(2)$ and the structure of tilting modules for $n=2$, we will determine the dimension, a set of generators and the decomposition of principal modules of $\mathrm{End}_{s_k(2,5)}(\Omega_k^{\otimes 5})$ and $\mathrm{End}_{u_k(2,5)}(\Omega_k^{\otimes 5})$, when $\Omega_k^{\otimes 5}$ is considered as a module of the infinitesimal q-Schur algebra and the little q-Schur algebra.

Key words: infinitesimal q-Schur algebra; little q-Schur algebra; Schur-Weyl duality

无穷小量子群 $u_k(n)$ 是由 Lusztig 在文献[1]中引入的一类重要的有限维 Hopf 代数，即经典模李代数理论中限制泛包络代数的量子化. q-Schur 代数 $U_k(n,r)$ 作为量子群的商代数，是联系量子群与 Hecke 代数的桥梁. 无穷小 q-Schur 代数 $s_k(n,r)$ 与小 q-Schur 代数 $u_k(n,r)$ 分别在文献[2-3]中被引入，对应于代数群 $\mathrm{gl}_k(n)$ 的闭子群 $\mathrm{gl}_k(n)_1T$ 与 $\mathrm{gl}_k(n)_1$，这里 $\mathrm{gl}_k(n)_1$ 和 T 分别为 $\mathrm{gl}_k(n)$ 的 Frobenius 核与一个极大环面. 因为 $\mathrm{End}_{u_k(n,r)}(\Omega_k^{\otimes r})=\mathrm{End}_{u_k(n)}(\Omega_k^{\otimes r})$，它将是研究无穷小情形下 Schur-Weyl 对偶的重要工具. 这里 k 是一个特征为零包含 l(奇数)次单位根 ε 的域，其中 $l\geqslant 3$(本文中只需考虑 $l=3$ 的情形). $u_k(2)$ 是 $\mathrm{gl}_k(2)$ 在 k 上的无穷小量子群，Ω_k 是 $\mathrm{gl}_k(2)$ 在 k 上的自然表示. 本文研究 $\mathrm{End}_{s_k(2,5)}(\Omega_k^{\otimes 5})$ 与 $\mathrm{End}_{u_k(2,5)}(\Omega_k^{\otimes 5})$ 的结构，给出它们的维数、一组生成元以及正则分解，希望在接下来的研究中得到更一般的结果.

1 无穷小 q-Schur 代数与小 q-Schur 代数

$\mathrm{gl}(2)$ 在域 $Q(v)$ (v 为一不定元)上的量子包络代数 $U=U(2)$ 有如下生成元：

$$E、F、K_i、K_i^{-1},\ 1\leqslant i\leqslant 2$$

满足如下关系式：

(1) $K_iK_j=K_jK_i$，$K_iK_i^{-1}=1$.

(2) $K_1E=vEK_1$，$K_2E=v^{-1}EK_2$.

(3) $K_1F=v^{-1}FK_1$，$K_2E=vFK_2$.

(4) $EF-FE=\dfrac{K_1K_2^{-1}-K_1^{-1}K_2}{v-v^{-1}}$.

$U(2)$ 为 Hopf 代数，其余乘在生成元上的定义为

$$\Delta(E)=E\otimes K_1K_2^{-1}+1\otimes E$$
$$\Delta(F)=F\otimes 1+K_1^{-1}K_2\otimes F$$
$$\Delta(K_i)=K_i\otimes K_i$$

设 $Z=\mathbf{Z}[v,v^{-1}]$，如文献[1]中所述，记 $U_Z(2)$ 为 $U(2)$ 中由 $E^{(m)}$、$F^{(m)}$、$K_i^{\pm 1}$ 以及 $\begin{bmatrix}K_i;0\\ t\end{bmatrix}$ 生成的 Z-子代数，其中 t、m 为一正整数，则

$$E^{(m)}=\frac{E^m}{[m]!},F^{(m)}=\frac{F^m}{[m]!},$$

$$\begin{bmatrix}K_i;c\\ t\end{bmatrix}=\prod_{s=1}^{t}\frac{K_iv^{c-s+1}-K_i^{-1}v^{-c+s-1}}{v^s-v^{-s}}$$

收稿日期：2016-05-27

第一作者：卞之豪(1990—)，男，博士生，主要研究方向为代数群、量子群及其表示理论. E-mail：zhihaobian@163.com

设k是一个特征为零包含l(奇数)次单位根ε的域,将v赋值为ε,则k可视为Z-模. 设$U_k(2)=U_Z(2)\otimes k$,仍用E、F、$K_i^{\pm 1}$表示它们在$U_k(2)$中的像. 记$\widetilde{u}_k(2)$为$U_k(2)$中由E、F、$K_i^{\pm 1}$生成的子代数,则$u_k(2)=\widetilde{u}_k(2)/(K_1^l-1,K_2^l-1)$即为$\mathrm{gl}(2)$在$k$上的无穷小量子群.

设Ω为基$\{\omega_i\mid 1\leqslant i\leqslant 2\}$张成的自由$Z$-模,记$\Omega_{Q(v)}=\Omega\otimes_Z Q(v)$,$\Omega_k=\Omega\otimes_Z k$,则$U(2)$在$\Omega_{Q(v)}$上有自然作用$E\omega_b=\delta_{2,b}\omega_{b-1}$,$F\omega_a=\delta_{1,a}\omega_{a+1}$,$K_a\omega_b=v^{\delta_{a,b}}\omega_b$. 上述模结构显然可诱导至域$k$,因此由$U_k(2)$的余乘可定义$U_k(2)$到$\Omega_k^{\otimes r}$上的作用,同时得到代数同态$\zeta_r:U_k(2)\to\mathrm{End}_k(\Omega_k^{\otimes 5})$,而$U_k(2,r):=\zeta_r(U_k(2))$即为域$k$上的 q-Schur 代数. 文献[3]中$u_k(2,r):=\zeta_r(\widetilde{u}_k(2))=\zeta_r(u_k(2))$即为小 q-Schur 代数.

记$e=\zeta_r(E)$, $f=\zeta_r(F)$, $k_i=\zeta_r(K_i)$,其中$1\leqslant i\leqslant 2$. 另设$k_\lambda=\prod_{i=1}^{2}\begin{bmatrix}K_i;0\\ \lambda_i\end{bmatrix}$,其中$\lambda\in\Lambda(2,r):=\{\lambda\in\mathbf{N}^2\mid\lambda_1+\lambda_2=r\}$. 对任意正整数$m$,记$Z_m=Z/mZ$,定义

$$\bar{\ }:Z^2\to(Z_l)^2$$

为由$\overline{(j_1,j_2)}=(\overline{j_1},\overline{j_2})$$(j_1,j_2\in\mathbf{Z})$决定的映射.

设$\overline{\Lambda(2,r)}=\{\bar{\lambda}\in(Z_l)^2\mid\lambda\in\Lambda(2,r)\}$,对$\bar{\lambda}\in(Z_l)^2$,定义

$$P_{\bar{\lambda}}=\begin{cases}\sum\limits_{\mu\in\Lambda(2,r),\bar{\mu}=\bar{\lambda}}k_\mu, & \bar{\lambda}\in\overline{\Lambda(2,r)}\\ 0, & \text{其他情形}\end{cases}$$

记$U_Z^{(0)}(2,r)$和$u_Z^{(0)}(2,r)$分别表示$U_Z(2,r)$和$u_Z(2,r)$零部分构成的子代数.

定理 1[3-4]　集合$\{k_\lambda\mid\lambda\in\Lambda(2,r)\}$ (resp. $\{P_{\bar{\lambda}}\mid\bar{\lambda}\in\overline{\Lambda(2,r)}\}$) 为$U_Z^{(0)}(2,r)$(resp. $u_Z^{(0)}(2,r)$)的一组完备的本原正交幂等元(因此也是一组基). 特别地,$1=\sum_{\lambda\in\Lambda(2,r)}k_\lambda=\sum_{\bar{\lambda}\in\overline{\Lambda(2,r)}}P_{\bar{\lambda}}$.

记$s_k(2,r)$为$U_k(2,r)$中由$u_k(2,r)$及$\begin{bmatrix}k_i;0\\ t\end{bmatrix}$生成的$k$-子代数,即是文献[2]中引入的无穷小 q-Schur 代数.

2　张量空间$\Omega_k^{\otimes 5}$的模结构

首先给出$n=2$时 tilting 模$T(\lambda)$与 Weyl 模$\Delta(\lambda)$(q-Schur 代数拟遗传性质中的标准模)的相关结果.

引理 1(文献[5]3.4 节)　设$\lambda=(\lambda_1,\lambda_2)$为支配权. 如果$\lambda_1-\lambda_2\leqslant l-1$或$\lambda_1-\lambda_2\equiv -1(\mathrm{mod}\ l)$(即$\lambda$为 Steinberg 权),则$T(\lambda)=\Delta(\lambda)$. 如果$\lambda_1-\lambda_2>l-1$且$\lambda_1-\lambda_2=bl+a$, $0\leqslant a<l-1$,则$T(\lambda)$有如下滤过:

$$0\subseteq\Delta(\lambda)\subseteq T(\lambda)$$

并且$T(\lambda)/\Delta(\lambda)\cong\Delta(\mu)$, $\mu=(\lambda_1-(a+1),\lambda_2+(a+1))$.

引理 2　设$\lambda=(\lambda_1,\lambda_2)$为支配权. 如果$\lambda_1-\lambda_2\leqslant l-1$或$\lambda_1-\lambda_2\equiv -1(\mathrm{mod}\ l)$(即$\lambda$为 Steinberg 权),则$\Delta(\lambda)=L(\lambda)$. 如果$\lambda_1-\lambda_2>l-1$且$\lambda_1-\lambda_2=bl+a$, $0\leqslant a<l-1$,则$\Delta(\lambda)$有如下滤过:

$$0\subseteq L(\mu)\subseteq\Delta(\lambda)$$

并且$\Delta(\lambda)/L(\mu)\cong L(\lambda)$, $\mu=(\lambda_1-(a+1),\lambda_2+(a+1))$.

证明　只需证明$\lambda_1-\lambda_2>l-1$且$\lambda_1-\lambda_2=bl+a$, $0\leqslant a<l-1$的情形. 由 Weyl 模$\Delta(\lambda)$的定义,其有如下形式特征标:

$$\mathrm{ch}\,\Delta(\lambda)=e(bl+a)+e(bl+a-2)+\cdots+e(-bl-a+2)+e(-bl-a)$$

由 Steinberg 张量积定理,$L(\lambda)\cong L(a)\otimes L(lb)$, $L(bl-a-2)\cong L(l-a-2)\otimes L(l(b-1))$,注意$L(\lambda)$即$L(\lambda_1-\lambda_2)$,这里将用划分参数化的单模与用一个整数参数化的单模混用. 因此$\mathrm{ch}\,L(\lambda)=\sum_{i=0}^{b}\sum_{j=0}^{a}e((b-2i)l+(a-2j))$, $\mathrm{ch}\,L(bl-a-2)=\sum_{i=0}^{b-1}\sum_{j=0}^{l-a-2}e((b-1-2i)l+(l-a-2-2j))$,不难看出$\mathrm{ch}\,\Delta(\lambda)=\mathrm{ch}\,L(\lambda)+\mathrm{ch}\,L(bl-a-2)$,引理成立.

作为$\mathrm{gl}_k(2)$的模,$\Omega_k\cong\Delta(1,0)\cong T(1,0)$,而由文献[5]中 3.3 节,tilting 模关于张量积运算封闭,因此作为$U_k(2,5)$的模,$\Omega_k^{\otimes 5}$有如下 tilting 模分解:

$$\Omega_k^{\otimes 5}\cong d_{(5,0)}T(5,0)+d_{(4,1)}T(4,1)+d_{(3,2)}T(3,2).$$

先取定$l=3$.

此时,由引理 1 和 2,$T(5,0)\cong L(5,0)$, $T(3,2)\cong L(3,2)$,而$T(4,1)$有如下滤过:

$$0\subseteq M_1\subseteq M_2\subseteq T(4,1)$$

其中,$M_1\cong L(3,2)$, $M_2\cong\Delta(4,1)$, $M_2/M_1\cong L(4,1)$, $T(4,1)/M_2\cong L(3,2)$.

引理 3　上述$\Omega_k^{\otimes 5}$的 tilting 模分解的重数$d_{(5,0)}=1$, $d_{(4,1)}=4$, $d_{(3,2)}=1$.

证明　由上述$T(\lambda)$的模结构,可得它们的特征标如下:

$$\mathrm{ch}\,T(5,0)=e((5,0))+e((4,1))+e((3,2))+e((2,3))+e((1,4))+e((0,5))$$

$$\mathrm{ch}\,T(4,1)=e((4,1))+2e((3,2))+2e((2,3))+$$

$e((1,4))$

$\mathrm{ch}\, T(3,2) = e((3,2)) + e((2,3))$

而 $\dim(\Omega_k^{\otimes 5})_{(a,5-a)} = \binom{5}{a}$,因此可依次计算

$$d_{(5,0)} = \binom{5}{5} = 1$$

$$d_{(4,1)} = \binom{5}{4} - d_{(5,0)}\dim(T(5,0))_{(4,1)} = 4$$

$$d_{(3,2)} = \binom{5}{3} - d_{(5,0)}\dim(T(5,0))_{(3,2)} - d_{(4,1)}\dim(T(4,1))_{(3,2)} = 1$$

引理4 上述 tilting 模 $T(\lambda)$间的代数同态的维数如下:

(1) $d_{(3,2)(\lambda)} = \dim \hom_{U_k(2,5)}(T(3,2),T(\lambda)) = \begin{cases} 1, & \lambda=(3,2) \text{ 或 } \lambda=(4,1) \\ 0, & \lambda=(5,0) \end{cases}$

$\hat{d}_{(3,2)(\lambda)} = \dim \hom_{s_k(2,5)}(T(3,2),T(\lambda)) = \dim \hom_{U_k(2,5)}(T(3,2),T(\lambda))$

$d^{(1)}_{(3,2)(\lambda)} = \dim \hom_{u_k(2,5)}(T(3,2),T(\lambda)) = \dim \hom_{U_k(2,5)}(T(3,2),T(\lambda))$

(2) $d_{(4,1)(\lambda)} = \dim \hom_{U_k(2,5)}(T(4,1),T(\lambda)) = \begin{cases} 1, & \lambda=(3,2) \\ 2, & \lambda=(4,1) \\ 0, & \lambda=(5,0) \end{cases}$

$\hat{d}_{(4,1)(\lambda)} = \dim \hom_{s_k(2,5)}(T(4,1),T(\lambda)) = \dim \hom_{U_k(2,5)}(T(4,1),T(\lambda))$

$d^{(1)}_{(4,1)(\lambda)} = \dim \hom_{u_k(2,5)}(T(4,1),T(\lambda)) = \dim \hom_{U_k(2,5)}(T(4,1),T(\lambda))$

(3) $d_{(5,0)(\lambda)} = \dim \hom_{U_k(2,5)}(T(5.0),T(\lambda)) = \begin{cases} 1, & \lambda=(5,0) \\ 0, & \lambda=(3,2) \text{ 或 } \lambda=(4,1) \end{cases}$

$\hat{d}_{(5,0)(\lambda)} = \dim \hom_{s_k(2,5)}(T(5.0),T(\lambda)) = \begin{cases} 2, & \lambda=(5,0) \\ 0, & \lambda=(3,2) \text{ 或 } \lambda=(4,1) \end{cases}$

$d^{(1)}_{(5,0)(\lambda)} = \dim \hom_{u_k(2,5)}(T(5.0),T(\lambda)) = \begin{cases} 4, & \lambda=(5,0) \\ 0, & \lambda=(3,2) \text{ 或 } \lambda=(4,1) \end{cases}$

证明 首先考虑 $T(\lambda)$作为$U_k(2,5)$-模的情形. 由上文的讨论,$T(5,0)\cong L(5,0)$,$T(3,2)\cong L(3,2)$,$\mathrm{soc}\, T(4,1)\cong L(3,2)$,$T(4,1)/\mathrm{Rad}\, T(4,1)\cong L(3,2)$,不难看出它们间的代数同态的维数如引理所述. 由 Steinberg 张量积定理,有

(1) $L(3,2)|_{s_k(2,5)} \cong \hat{L}_1(3,2)$,$L(3,2)|_{u_k(2,5)} \cong L_1(3,2)$.

(2) $L(4,1)|_{s_k(2,5)} \cong L(0)\otimes L(3)|_{s_k(2,5)} \cong k_{(3)} \oplus k_{(-3)}$,$L(4,1)|_{u_k(2,5)} \cong L(0)\otimes L(3)|_{u_k(2,5)} \cong k_{(0)} \oplus k_{(0)}$.

(3) $L(5,0)|_{s_k(2,5)} \cong \hat{L}_1(5,0) \oplus \hat{L}_1(2,3)$,$L(5,0)|_{u_k(2,5)} \cong L_1(2) \oplus L_1(2)$.

这里$\hat{L}_1(\lambda)$和$L_1(\lambda)$分别表示 $\mathrm{gl}_k(2)_1 T$ 和 $\mathrm{gl}_k(2)_1$对应的单模(参看文献[5]第三章),$k_{(\alpha)}$、$\alpha\in\mathbf{Z}$表示权为α的一维模空间. 同理可得 $T(\lambda)$限制在 $s_k(2,5)$,$u_k(2,5)$上的情形. 因此,可以分别得到 $\mathrm{End}_{U_k(2,5)}(\Omega_k^{\otimes 5})$、$\mathrm{End}_{s_k(2,5)}(\Omega_k^{\otimes 5})$与 $\mathrm{End}_{u_k(2,5)}(\Omega_k^{\otimes 5})$的维数如下:

(1) $\dim \mathrm{End}_{U_k(2,5)}(\Omega_k^{\otimes 5}) = \sum\limits_{\lambda,\mu\in\Lambda(2,5)} d_\lambda d_\mu d_{(\lambda)(\mu)} = 42$.

(2) $\dim \mathrm{End}_{s_k(2,5)}(\Omega_k^{\otimes 5}) = \sum\limits_{\lambda,\mu\in\Lambda(2,5)} d_\lambda d_\mu \hat{d}_{(\lambda)(\mu)} = 43$.

(3) $\dim \mathrm{End}_{u_k(2,5)}(\Omega_k^{\otimes 5}) = \sum\limits_{\lambda,\mu\in\Lambda(2,5)} d_\lambda d_\mu d^{(1)}_{(\lambda)(\mu)} = 45$.

当$l>5$时,$U_k(2,5)=s_k(2,5)=u_k(2,5)$,此时 $\mathrm{End}_{U_k(2,5)}(\Omega_k^{\otimes 5}) = \mathrm{End}_{u_k(2,5)}(\Omega_k^{\otimes 5})$同构于 Hecke 代数 $H(5)$模去其作用于张量空间 $\Omega_k^{\otimes 5}$ 上的核后所得的商代数,并且 $T(5,0)\cong L(5,0)$,$T(4,1)\cong L(4,1)$,$T(3,2)\cong L(3,2)$. 与 $l=3$ 时的讨论相类似,有$d_{(5,0)}=1$,$d_{(4,1)}=4$,$d_{(3,2)}=5$,并且 $d_{(\lambda)(\mu)}=\delta_{(\lambda)(\mu)}$. 因此可得 $\mathrm{End}_{u_k(2,5)}(\Omega_k^{\otimes 5})$ 半单且 $\dim \mathrm{End}_{u_k(2,5)}(\Omega_k^{\otimes 5})=42$.

当$l=5$时,由上述引理,$T(4,1)\cong L(4,1)$,$T(3,2)\cong L(3,2)$,而$T(5,0)$有如下滤过:

$$0\subseteq M_1\subseteq M_2\subseteq T(5,0)$$

其中$M_1\cong L(4,1)$,$M_2\cong\Delta(5,0)$. 同理可得 $d_{(5,0)}=1$,$d_{(4,1)}=3$,$d_{(3,2)}=5$ 并且 $d_{(\lambda)(\mu)}=\hat{d}_{(\lambda)(\mu)}=d^{(1)}_{(\lambda)(\mu)}$. 因此可得 $\mathrm{End}_{U_k(2,5)}(\Omega_k^{\otimes 5}) = \mathrm{End}_{u_k(2,5)}(\Omega_k^{\otimes 5})$ 并且 $\dim \mathrm{End}_{u_k(2,5)}(\Omega_k^{\otimes 5})=42$. 事实上,对任意的满足 $r<2l-1$ 的 r 和 l,均有 $\mathrm{End}_{U_k(2,r)}(\Omega_k^{\otimes r}) = \mathrm{End}_{u_k(2,r)}(\Omega_k^{\otimes r})$.

以下两节取定 $l=3$.

3 $\mathrm{End}_{s_k(2,5)}(\Omega_k^{\otimes 5})$的结构

记$\hat{T}(\lambda):=T(\lambda)|_{s_k(2,5)}$,$\Omega_k^{\otimes 5}$ 作为 $s_k(2,5)$的表示空间有如下分解:

$$\Omega_k^{\otimes 5}|_{s_k(2,5)} = \bigoplus_{i=1}^{4}\hat{T}(4,1)^{(i)} \oplus \hat{T}(3,2) \oplus \hat{L}_1(5,0) \oplus \hat{L}_1(2,3)$$

定义 1　设 M 为 $\Omega_k^{\otimes 5}|_{s_k(2,5)}$ 在上述分解中的某一直和项，分别定义 $\mathrm{End}_{s_k(2,5)}(\Omega_k^{\otimes 5})$ 中元素 $s_{(4,1)(i)}$ $(1\leqslant i\leqslant 4)$、$s_{(3,2)}$、$\eta_{(5,0)(1)}$、$\eta_{(5,0)(2)}$、$t_{(4,1)(3,2)}$、$t_{(3,2)(4,1)}$，如下所示：

(1) $s_{(4,1)(i)}(M)=$

$$\begin{cases}\hat{T}(4,1)^{(i+1)}, & M=\hat{T}(4,1)^{(i)},1\leqslant i\leqslant 3\\ \hat{T}(4,1)^{(1)}, & M=\hat{T}(4,1)^{(4)},i=4\\ 0, & \text{其他情形}\end{cases}$$

(2) $s_{(3,2)}(M)=\begin{cases}\hat{T}(3,2), & M=\hat{T}(3,2)\\ 0, & \text{其他情形}\end{cases}$

(3) $\eta_{(5,0)(j)}(M)=$

$$\begin{cases}\hat{L}_1(5,0), & M=\hat{L}_1(5,0),j=1\\ \hat{L}_1(2,3), & M=\hat{L}_1(2,3),j=2\\ 0, & \text{其他情形}\end{cases}$$

(4) $t_{(4,1)(3,2)}(M)=\begin{cases}\hat{T}(3,2), & M=\hat{T}(4,1)^{(1)}\\ 0, & \text{其他情形}\end{cases}$

(5) $t_{(3,2)(4,1)}(M)=\begin{cases}\hat{T}(4,1)^{(1)}, & M=\hat{T}(3,2)\\ 0, & \text{其他情形}\end{cases}$

为方便之后的讨论，定义 $s_{(4,1)(i+4)}:=s_{(4,1)(i)}$，$s_{(4,1)}^{(i)}=s_{(4,1)(i+3)}s_{(4,1)(i+2)}s_{(4,1)(i+1)}s_{(4,1)(i)}$.

定理 2　定义 1 中给出的元素为$\mathrm{End}_{s_k(2,5)}(\Omega_k^{\otimes 5})$的一组生成元.

证明　记 $S_{(4,1)}$ 为由 $s_{(4,1)(i)}$ $(1\leqslant i\leqslant 4)$ 生成的子代数. 由引理 4 及定义 1，直和项 $\hat{T}(4,1)^{(i)}$ 到 $\hat{T}(4,1)^{(j)}$ 间的同构形如 $ks_{(4,1)(j-1)}\cdots s_{(4,1)(i+1)}s_{(4,1)(i)}$，$\hat{T}(4,1)^{(i)}$ 到 $\hat{T}(3,2)$ 的满同态形如 $kt_{(4,1)(3,2)}s_{(4,1)(4)}\cdots s_{(4,1)(i+1)}s_{(4,1)(i)}$，$\hat{T}(4,1)^{(i)}$ 到 soc $\hat{T}(4,1)^{(j)}$ 满同态形如 $ks_{(4,1)(j-1)}\cdots s_{(4,1)(i+1)}s_{(4,1)(1)}t_{(3,2)(4,1)}t_{(4,1)(3,2)}s_{(4,1)(4)}\cdots s_{(4,1)(i+1)}s_{(4,1)(i)}$. 因此，$\hom_{s_k(2,5)}(\bigoplus_{i=1}^{4}\hat{T}(4,1)^{(i)},\Omega_k^{\otimes 5})$ 中的元素均可由形如 s、$t_{(4,1)(3,2)}s$、$st_{(3,2)(4,1)}t_{(4,1)(3,2)}s$ 的元素线性张成，其中 $s\in S_{(4,1)}$.

同理，$\hom_{s_k(2,5)}(\hat{T}(3,2),\Omega_k^{\otimes 5})$中元素均可由形如 $s_{(3,2)}$ 和 $st_{(3,2)(4,1)}$ 的元素线性张成，$s\in S$. 因此 $\{s_{(4,1)(i)}(1\leqslant i\leqslant 4),s_{(3,2)},\eta_{(5,0)(1)},\eta_{(5,0)(2)},t_{(4,1)(3,2)},t_{(3,2)(4,1)}\}$为 $\mathrm{End}_{s_k(2,5)}(\Omega_k^{\otimes 5})$的一组生成元.

同样由上述生成元的定义，可得下述引理.

引理 5　$\mathrm{End}_{s_k(2,5)}(\Omega_k^{\otimes 5})$的上述生成元满足如下关系式，其中 x、y 为生成元中某一元素.

(1) $s_{(4,1)(i)}s_{(4,1)}^{(i)}=s_{(4,1)(i)}$；$s_{(4,1)(i)}s_{(4,1)(j)}=0$，$i\neq j+1$；$xs_{(4,1)(i)}=0$，$x\neq t_{(4,1)(3,2)}$，$i\neq 4$；$s_{(4,1)(i)}y=0$，$y\neq t_{(3,2)(4,1)}$，$i\neq 1$.

(2) $s_{(3,2)}t_{(4,1)(3,2)}=t_{(4,1)(3,2)}s_{(4,1)}^{(1)}=t_{(4,1)(3,2)}$；$xt_{(4,1)(3,2)}=0$，$x\neq s_{(3,2)}$，$t_{(3,2)(4,1)}$；$t_{(4,1)(3,2)}y=0$，$y\neq s_{(4,1)(4)}$.

(3) $t_{(3,2)(4,1)}s_{(3,2)}=s_{(4,1)}^{(1)}t_{(3,2)(4,1)}=t_{(3,2)(4,1)}$；$xt_{(3,2)(4,1)}=0$，$x\neq s_{(4,1)(1)}$；$t_{(3,2)(4,1)}y=0$，$y\neq s_{(3,2)}$，$t_{(4,1)(3,2)}$.

(4) $(s_{(3,2)})^2=s_{(3,2)}$；$xs_{(3,2)}=0$，$x\neq s_{(3,2)}$，$t_{(3,2)(4,1)}$；$s_{(3,2)}y=0$，$y\neq s_{(3,2)}$，$t_{(4,1)(3,2)}$.

(5) $(\eta_{(5,0)(j)})^2=\eta_{(5,0)(j)}$；$x\eta_{(5,0)(j)}=\eta_{(5,0)(j)}x=0$，$x\neq\eta_{(5,0)(j)}$.

由定义 1 及上述关系式，有 $(s_{(4,1)}^{(i)})^2=s_{(4,1)}^{(i)}$，$(s_{(3,2)})^2=s_{(3,2)}$，$(\eta_{(5,0)(j)})^2=\eta_{(5,0)(j)}$，并且 $s_{(4,1)}^{(i)}\in\hom_{s_k(2,5)}(\hat{T}(4,1)^{(i)},\hat{T}(4,1)^{(i)})$，$s_{(3,2)}\in\hom_{s_k(2,5)}(\hat{T}(3,2),\hat{T}(3,2))$，以及 $\eta_{(5,0)(1)}\in\hom_{s_k(2,5)}(\hat{L}_1(5,0),\hat{L}_1(5,0))$，$\eta_{(5,0)(2)}\in\hom_{s_k(2,5)}(\hat{L}_1(2,3),\hat{L}_1(2,3))$，因此 $\mathrm{End}_{s_k(2,5)}(\Omega_k^{\otimes 5})$ 有如下极小单位分解：

$$1=\sum_{i=1}^{4}s_{(4,1)}^{(i)}+s_{(3,2)}+\eta_{(5,0)(1)}+\eta_{(5,0)(2)}$$

记 $P_{(4,1)}^{(i)}=\mathrm{End}_{s_k(2,5)}(\Omega_k^{\otimes 5})\cdot s_{(4,1)}^{(i)}$，$P_{(3,2)}=\mathrm{End}_{s_k(2,5)}(\Omega_k^{\otimes 5})\cdot s_{(3,2)}$，$P_{(5,0)}^{(j)}=\mathrm{End}_{s_k(2,5)}(\Omega_k^{\otimes 5})\cdot\eta_{(5,0)(j)}$，同时 $R_{(\lambda)}^{(i)}=\mathrm{Rad}\,P_{(\lambda)}^{(i)}$，$U_{(\lambda)}^{(i)}=P_{(\lambda)}^{(i)}/R_{(\lambda)}^{(i)}$. 注意若 λ 不是 Steinberg 权，则 $P_{(\lambda)}^{(i)}\cong P_{(\lambda)}^{(j)}$.

定理 3　$\mathrm{End}_{s_k(2,5)}(\Omega_k^{\otimes 5})$的射影模 $P_{(4,1)}^{(1)}$、$P_{(3,2)}$、$P_{(5,0)}^{(1)}$、$P_{(5,0)}^{(2)}$的结构如下：

(1) $(R_{(4,1)}^{(1)})^3=0$，$(R_{(4,1)}^{(1)})^2=\mathrm{span}_k\{t_{(3,2)(4,1)}t_{(4,1)(3,2)},s_{(4,1)(1)}t_{(3,2)(4,1)}t_{(4,1)(3,2)},s_{(4,1)(2)}s_{(4,1)(1)}t_{(3,2)(4,1)}t_{(4,1)(3,2)},s_{(4,1)(3)}s_{(4,1)(2)}s_{(4,1)(1)}t_{(3,2)(4,1)}t_{(4,1)(3,2)}\}$，$R_{(4,1)}^{(1)}=\mathrm{span}_k\{t_{(4,1)(3,2)}\}\oplus(R_{(4,1)}^{(1)})^2$，$P_{(4,1)}^{(1)}=\mathrm{span}_k\{s_{(4,1)}^{(1)},s_{(4,1)(1)},s_{(4,1)(2)}s_{(4,1)(1)},s_{(4,1)(3)}s_{(4,1)(2)}s_{(4,1)(1)}\}\oplus R_{(4,1)}^{(1)}$，并且$(R_{(4,1)}^{(1)})^2\cong U_{(4,1)}^{(1)}$，$R_{(4,1)}^{(1)}/(R_{(4,1)}^{(1)})^2\cong U_{(3,2)}^{(1)}$.

(2) $(R_{(3,2)}^{(1)})^2=0$，$R_{(3,2)}^{(1)}=\mathrm{span}_k\{t_{(3,2)(4,1)},s_{(4,1)(1)}t_{(3,2)(4,1)},s_{(4,1)(2)}s_{(4,1)(1)}t_{(3,2)(4,1)},s_{(4,1)(3)}s_{(4,1)(2)}s_{(4,1)(1)}t_{(3,2)(4,1)}\}$，$P_{(3,2)}^{(1)}=\mathrm{span}_k\{s_{(3,2)}\}\oplus R_{(3,2)}^{(1)}$，并且 $R_{(3,2)}^{(1)}\cong U_{(4,1)}^{(1)}$.

(3) $R_{(5,0)}^{(1)}=R_{(5,0)}^{(2)}=0$，$P_{(5,0)}^{(1)}=\mathrm{span}_k\{\eta_{(5,0)(1)}\}$，$P_{(5,0)}^{(2)}=\mathrm{span}_k\{\eta_{(5,0)(2)}\}$.

证明　给出定理 3(1)的证明，同理可得定理 3

(2)和(3). $s_{(4,1)}^{(1)}$作为左$\mathrm{End}_{s_k(2,5)}(\Omega_k^{\otimes 5})$模$P_{(4,1)}^{(1)}$的生成元,由定义1和引理5、定理3(1)中张成$P_{(4,1)}^{(1)}$的元素穷举了所有可由$s_{(4,1)}^{(1)}$生成的$\mathrm{End}_{s_k(2,5)}(\Omega_k^{\otimes 5})$的基元素(它是$\hom_{s_k(2,5)}(\hat{T}(4,1)^{(1)},\Omega_k^{\otimes 5})$的一组基).由引理5(1),可得

$\{s_{(4,1)}^{(1)}, s_{(4,1)(1)}, s_{(4,1)(2)}s_{(4,1)(1)}, s_{(4,1)(3)}s_{(4,1)(2)}s_{(4,1)(1)}\}$

中的元素可以相互生成,同理

$\{t_{(3,2)(4,1)}t_{(4,1)(3,2)}, s_{(4,1)(1)}t_{(3,2)(4,1)}t_{(4,1)(3,2)},$
$s_{(4,1)(2)}s_{(4,1)(1)}t_{(3,2)(4,1)}t_{(4,1)(3,2)},$
$s_{(4,1)(3)}s_{(4,1)(2)}s_{(4,1)(1)}t_{(3,2)(4,1)}t_{(4,1)(3,2)}\}$

中的元素也可以相互生成.而由引理5(2)和(3), $s_{(4,1)}^{(1)}$可生成$t_{(4,1)(3,2)}$, $t_{(4,1)(3,2)}$可生成$t_{(3,2)(4,1)}t_{(4,1)(3,2)}$,反之不行,因此分别得到$(R_{(4,1)}^{(1)})^2$、$R_{(4,1)}^{(1)}$、$P_{(4,1)}^{(1)}$如定理所述的一组基.余下的结论就容易得到了.

4 $\mathrm{End}_{u_k(2,5)}(\Omega_k^{\otimes 5})$的结构

$r=5$(r较小)时,$\mathrm{End}_{u_k(2,5)}(\Omega_k^{\otimes 5})$与$\mathrm{End}_{s_k(2,5)}(\Omega_k^{\otimes 5})$的结构十分类似,仿照上一节的内容给出结论,证明是完全类似的.

记$T^{(1)}(\lambda):=T(\lambda)|_{u_k(2,5)}$,$\Omega_k^{\otimes 5}$作为$u_k(2,5)$的表示空间有如下分解:

$$\Omega_k^{\otimes 5}|_{u_k(2,5)}=\bigoplus_{i=1}^{4}T^{(1)}(4,1)^{(i)}\oplus T^{(1)}(3,2)\oplus L_1(2)^{(1)}\oplus L_1(2)^{(2)}$$

定义2 设M为$\Omega_k^{\otimes 5}|_{u_k(2,5)}$的上述分解中的某一直和项,分别定义$\mathrm{End}_{u_k(2,5)}(\Omega_k^{\otimes 5})$中元素$s_{(4,1)(i)}$ $(1\leqslant i\leqslant 4)$、$s_{(3,2)}$、$\theta_{(5,0)(1)}$、$\theta_{(5,0)(2)}$、$t_{(4,1)(3,2)}$、$t_{(3,2)(4,1)}$,如下所示:

(1) $s_{(4,1)(i)}(M)=$
$$\begin{cases}T^{(1)}(4,1)^{(i+1)}, & M=T^{(1)}(4,1)^{(i)},1\leqslant i\leqslant 3\\ T^{(1)}(4,1)^{(1)}, & M=T^{(1)}(4,1)^{(4)},i=4\\ 0, & \text{其他情形}\end{cases}$$

(2) $$s_{(3,2)}(M)=\begin{cases}T^{(1)}(3,2), & M=T^{(1)}(3,2)\\ 0, & \text{其他情形}\end{cases}$$

(3) $\theta_{(5,0)(j)}(M)=$
$$\begin{cases}L_1(2)^{(2)}, & M=L_1(2)^1,j=1\\ L_1(2)^{(1)}, & M=L_1(2)^2,j=2\\ 0, & \text{其他情形}\end{cases}$$

(4) $t_{(4,1)(3,2)}(M)=$
$$\begin{cases}T^{(1)}(3,2), & M=T^{(1)}(4,1)^{(1)}\\ 0, & \text{其他情形}\end{cases}$$

(5) $t_{(3,2)(4,1)}(M)=$
$$\begin{cases}T^{(1)}(4,1)^{(1)}, & M=T^{(1)}(3,2)\\ 0, & \text{其他情形}\end{cases}$$

附注1 注意事实上$\{s_{(4,1)(i)}(1\leqslant i\leqslant 4), s_{(3,2)}, t_{(4,1)(3,2)}, t_{(3,2)(4,1)}\}\subset\mathrm{End}_{U_k(2,5)}(\Omega_k^{\otimes 5})$,可直接验证这些元素与$e^{(3)}$、$f^{(3)}$、$k_\lambda$交换.

类似地定义$\theta_{(5,0)(j+2)}=\theta_{(5,0)(j)}$, $\theta_{(5,0)}^{(j)}=\theta_{(5,0)(j+1)}\theta_{(5,0)(j)}$.

引理6 定义2中给出的元素为$\mathrm{End}_{u_k(2,5)}(\Omega_k^{\otimes 5})$的一组生成元且满足如下关系式:

(1) $s_{(4,1)(i)}s_{(4,1)}^{(i)}=s_{(4,1)(i)}$; $s_{(4,1)(i)}s_{(4,1)(j)}=0, i\neq j+1$; $xs_{(4,1)(i)}=0, x\neq t_{(4,1)(3,2)}, i\neq 4$; $s_{(4,1)(i)}y=0, y\neq t_{(3,2)(4,1)}, i\neq 1$.

(2) $s_{(3,2)}t_{(4,1)(3,2)}=t_{(4,1)(3,2)}s_{(4,1)}^{(1)}=t_{(4,1)(3,2)}$; $xt_{(4,1)(3,2)}=0$, $x\neq s_{(3,2)}, t_{(3,2)(4,1)}$; $t_{(4,1)(3,2)}y=0$, $y\neq s_{(4,1)(4)}$.

(3) $t_{(3,2)(4,1)}s_{(3,2)}=s_{(4,1)}^{(1)}t_{(3,2)(4,1)}=t_{(3,2)(4,1)}$; $xt_{(3,2)(4,1)}=0$, $x\neq s_{(4,1)(1)}$; $t_{(3,2)(4,1)}y=0$, $y\neq s_{(3,2)}, t_{(4,1)(3,2)}$.

(4) $(s_{(3,2)})^2=s_{(3,2)}$; $xs_{(3,2)}=0$, $x\neq s_{(3,2)}, t_{(3,2)(4,1)}$; $s_{(3,2)}y=0, y\neq s_{(3,2)}, t_{(4,1)(3,2)}$.

(5) $\theta_{(5,0)(j)}\theta_{(5,0)}^{(j)}=\theta_{(5,0)(j)}$; $x\theta_{(5,0)(j)}=\theta_{(5,0)(j)}x=0, x\neq\theta_{(5,0)(j+1)}$.

由定义2及上述关系式,$\mathrm{End}_{u_k(2,5)}(\Omega_k^{\otimes 5})$有如下极小单位分解:

$$1=\sum_{i=1}^{4}s_{(4,1)}^{(i)}+s_{(3,2)}+\theta_{(5,0)(1)}+\theta_{(5,0)(2)}$$

仍记$P_{(4,1)}^{(i)}=\mathrm{End}_{s_k(2,5)}(\Omega_k^{\otimes 5})\cdot s_{(4,1)}^{(i)}$, $P_{(3,2)}=\mathrm{End}_{s_k(2,5)}(\Omega_k^{\otimes 5})\cdot s_{(3,2)}$, $P_{(5,0)}^{(j)}=\mathrm{End}_{s_k(2,5)}(\Omega_k^{\otimes 5})\cdot\theta_{(5,0)(j)}$.同时$R_{(\lambda)}^{(i)}=\mathrm{Rad}\,P_{(\lambda)}^{(i)}$, $U_{(\lambda)}^{(i)}=P_{(\lambda)}^{(i)}/R_{(\lambda)}^{(i)}$.注意$P_{(\lambda)}^{(i)}\cong P_{(\lambda)}^{(j)}$.

定理4 $\mathrm{End}_{u_k(2,5)}(\Omega_k^{\otimes 5})$的射影模$P_{(4,1)}^{(1)}$、$P_{(3,2)}$、$P_{(5,0)}^{(1)}$结构如下:

(1) $(R_{(4,1)}^{(1)})^3=0$, $(R_{(4,1)}^{(1)})^2=\mathrm{span}_k\{t_{(3,2)(4,1)}t_{(4,1)(3,2)}, s_{(4,1)(1)}t_{(3,2)(4,1)}t_{(4,1)(3,2)}, s_{(4,1)(2)}s_{(4,1)(1)}t_{(3,2)(4,1)}t_{(4,1)(3,2)}, s_{(4,1)(3)}s_{(4,1)(2)}s_{(4,1)(1)}t_{(3,2)(4,1)}t_{(4,1)(3,2)}\}$, $R_{(4,1)}^{(1)}=\mathrm{span}_k\{t_{(4,1)(3,2)}\}\oplus(R_{(4,1)}^{(1)})^2$; $P_{(4,1)}^{(1)}=\mathrm{span}_k\{s_{(4,1)}^{(1)}, s_{(4,1)(1)}, s_{(4,1)(2)}s_{(4,1)(1)}, s_{(4,1)(3)}s_{(4,1)(2)}s_{(4,1)(1)}\}\oplus R_{(4,1)}^{(1)}$,并且$(R_{(4,1)}^{(1)})^2\cong U_{(4,1)}^{(1)}$, $R_{(4,1)}^{(1)}/(R_{(4,1)}^{(1)})^2\cong U_{(3,2)}^{(1)}$.

(2) $(R_{(3,2)}^{(1)})^2=0$, $R_{(3,2)}^{(1)}=\mathrm{span}_k\{t_{(3,2)(4,1)}, s_{(4,1)(1)}t_{(3,2)(4,1)}, s_{(4,1)(2)}s_{(4,1)(1)}t_{(3,2)(4,1)}, s_{(4,1)(3)}s_{(4,1)(2)}s_{(4,1)(1)}t_{(3,2)(4,1)}\}$, $P_{(3,2)}^{(1)}=\mathrm{span}_k\{s_{(3,2)}\}\oplus R_{(3,2)}^{(1)}$,并且$R_{(3,2)}^{(1)}\cong U_{(4,1)}^{(1)}$.

(3) $R_{(5,0)}^{(1)}=0$, $P_{(5,0)}^{(1)}=\mathrm{span}_k\{\theta_{(5,0)(1)}, \theta_{(5,0)}^{(1)}\}$.

参考文献：

[1] LUSZTIG G. Finite dimensional Hopf algebras arising from quantized universal enveloping algebras [J]. Journal of American Mathematical Society, 1990, 3: 257.

[2] COZ A G. On some applications of infinitesimal methods to quantum groups and related algebras [D]. London: London Univeristy, 1997.

[3] DU J, FU Q, Wang J-P. Infinitesimal quantum gl_n and little q-Schur algebras [J]. Journal of Algebra, 2005, 287:199.

[4] DOTY S, GIAQUINTO A. Presenting Schur algebras [J]. International Mathematics Research Notices, 2002, 36: 1907.

[5] DONKIN S. The q-Schur algebra: London mathematical society lecture note series 253[M]. New York: Cambridge University Press, 2010.

(上接第 921 页)

[4] BLANGERO A, GAVEAU V, LUAUTE J, *et al*. A hand and a field effect in on-line motor control in unilateral optic ataxia [J]. Cortex, 2008, 44(5): 560.

[5] DESMURGET M, EPSTEIN C M, TURNER R S, *et al*. Role of the posterior parietal cortex in updating reaching movements to a visual target [J]. Nature Neuroscience, 1999, 2(6): 563.

[6] HIMMELBACH M, KARNATH H, PERENIN M, *et al*. A general deficit of the 'automatic pilot' with posterior parietal cortex lesions? [J]. Neuropsychologia, 2006, 44(13): 2749.

[7] BRENNER E, SMEETS J B. Colour vision can contribute to fast corrections of arm movements [J]. Experimental Brain Research, 2004, 158(3): 302.

[8] CAMERON B D, CRESSMAN E K, FRANKS I M, *et al*. Cognitive constraint on the 'automatic pilot' for the hand: movement intention influences the hand's susceptibility to involuntary online corrections [J]. Consciousness and Cognition, 2009, 18(3): 646.

[9] DAY B L, LYON I N. Voluntary modification of automatic arm movements evoked by motion of a visual target [J]. Experimental Brain Research, 2000, 130(2): 159.

[10] MCINTOSH R D, MULROUE A, BROCKMOLE J R. How automatic is the hand's automatic pilot? [J]. Experimental Brain Research, 2010, 206(3): 257.

[11] BRENNER E, SMEETS J B. Two eyes in action [J]. Experimental Brain Research, 2006, 170(3): 302.

[12] VAN MIERLO C M, LOUW S, SMEETS J B, *et al*. Slant cues are processed with different latencies for the online control of movement [J]. Journal of Vision, 2009, 9(3): 21.

[13] SONG Y, SUN Y, ZENG J, *et al*. Automatic correction of hand pointing in stereoscopic depth[J]. Scientific Reports, 2014, 4: 7444.

[14] CALTON J L, DICKINSON A R, SNYDER L H. Non-spatial, motor-specific activation in posterior parietal cortex [J]. Nature Neuroscience, 2002, 5(6): 580.

[15] RODRÍGUEZ-FORNELLS A, KURZBUCH A R, MÜNTE T F. Time course of error detection and correction in humans: neurophysiological evidence [J]. Journal of Neuroscience the Official Journal of the Society for Neuroscience, 2002, 22(22): 9990.

[16] SPÜLER M, NIETHAMMER C. Error-related potentials during continuous feedback: using EEG to detect errors of different type and severity [J]. Frontiers in Human Neuroscience, 2015, 9:155.

[17] 侯文生，张阳，李卫娜，等. 手指自主按键运动中脑磁图信号的相干性分析[J]. 山东大学学报(理学版), 2011, 46(1):1.
HOU Wensheng, ZHANG Yang, LI Weina, *et al*. Coherence analysis of magnetoencephalography signals related to self-paced finger press [J]. Journal of Shandong University (Natural Science), 2011, 46(1):1.

[18] WASCHER E, HOFFMANN S, SANGER J, *et al*. Visuo-spatial processing and the N1 component of the ERP [J]. Psychophysiology, 2009, 46(6):1270.

[19] ROSLER F, SUTTON S, JOHNSON R J, *et al*. Endogenous ERP components and cognitive constructs [J]. Electroencephalography and Clinical Neurophysiology Supplement, 1986, 38:51.

[20] PICTON T W. The P300 wave of the human event-related potential [J]. Journal of Clinical Neurophysiology, 1992, 9(4):456.